# SECTION 5

# GEOCENTRIC SPACE-FIXED POSITION, VELOCITY, AND ACCELERATION VECTORS OF TRACKING STATION

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## 5.1 INTRODUCTION

This section gives the extensive formulation for the geocentric space-fixed position, velocity, and acceleration vectors of a fixed tracking station on Earth. These vectors are referred to the celestial reference frame defined by the planetary ephemeris (the planetary ephemeris frame, PEF).

Section 5.2 gives the formulation for the Earth-fixed position vector $\mathbf{r}_b$ of a fixed tracking station on Earth. The rectangular components of this vector are referred to the true pole, prime meridian, and equator of date. The formulation includes terms for the coordinates of the tracking station (referred to the mean pole, prime meridian, and equator of 1903.0), the Earth-fixed velocity components of the tracking station due to plate motion, polar motion, solid Earth tides, ocean loading, and the pole tide. Section 5.3 gives the formulation for the Earth-fixed to space-fixed transformation matrix $T_E$ and its first and second time derivatives with respect to coordinate time ET. The matrix $T_E$ includes the frame-tie rotation matrix, which relates the radio frame RF (a particular celestial reference frame maintained by the International Earth Rotation Service, IERS) and the PEF. Without the frame-tie rotation matrix, the matrix $T_E$ would rotate to the RF. With the frame-tie rotation matrix included, $T_E$ rotates to the PEF. Program PV uses an alternate version of $T_E$ which rotates from the Earth-fixed coordinate system referred to the mean pole, prime meridian, and equator of 1903.0. This version of $T_E$ is obtained from the version used in Regres by adding rotations through the polar motion angles $X$ and $Y$.

Section 5.4 uses $\mathbf{r}_b$ and $T_E$ and its time derivatives to calculate the geocentric space-fixed position, velocity, and acceleration vectors of a fixed tracking station on Earth, referred to the PEF. When the ODP uses the Solar-System barycentric space-time frame of reference, the geocentric space-fixed position vector of the tracking station is transformed from the local geocentric space-time frame of reference to the Solar-System barycentric space-time frame of reference using Eq. (4–10).

The partial derivatives of the geocentric space-fixed position vector of the tracking station with respect to Earth-fixed station coordinates and other solve-for parameters are given in Section 5.5.

The time argument for calculating the Earth-fixed position vector $\mathbf{r}_\mathrm{b}$ and the Earth-fixed to space-fixed transformation matrix $T_\mathrm{E}$ and its time derivatives is coordinate time ET in the Solar-System barycentric or local geocentric space-time frame of reference. For a spacecraft light-time solution, the time argument will be the reception time $t_3$(ET) in coordinate time ET at the receiving station on Earth or the transmission time $t_1$(ET) at the transmitting station on Earth. For a quasar light-time solution, the time argument will be the reception time $t_1$(ET) of the quasar wavefront at receiving station 1 on Earth or the reception time $t_2$(ET) of the wavefront at receiving station 2 on Earth.

## 5.2 EARTH-FIXED POSITION VECTOR OF TRACKING STATION

The Earth-fixed position vector $\mathbf{r}_\mathrm{b}$ of a fixed tracking station on Earth, with rectangular components referred to the true pole, prime meridian, and equator of date, is given by the following sum of terms:

$$\mathbf{r}_\mathrm{b} = \begin{bmatrix} x_\mathrm{b} \\ y_\mathrm{b} \\ z_\mathrm{b} \end{bmatrix} = \alpha\,\mathbf{r}_{\mathrm{b}_0} + \Delta\mathbf{r}_{\mathrm{b}_0} + \dot{\mathbf{r}}_\mathrm{b}\left(t - t_0\right) + \mathbf{r}_\mathrm{O} \qquad \mathrm{km} \qquad (5\text{–}1)$$
$$+ \Delta\mathbf{r}_\mathrm{PM} + \Delta\mathbf{r}_\mathrm{SET} + \Delta\mathbf{r}_\mathrm{OL} + \Delta\mathbf{r}_\mathrm{PT}$$

Subsections 5.2.1 to 5.2.8 correspond to the eight terms of Eq. (5–1). Each section defines the corresponding term of Eq. (5–1) and gives the formulation for computing it.

### 5.2.1 1903.0 POSITION VECTOR OF TRACKING STATION OR NEARBY SURVEY BENCHMARK

The first term of Eq. (5–1) contains the geocentric Earth-fixed position vector $\mathbf{r}_{\mathrm{b}_0}$ of the tracking station or a nearby survey benchmark, with rectangular components referred to the mean pole, prime meridian, and equator of 1903.0. The station location is the intersection of the two axes of the antenna. If the axes do not intersect, it is on the primary axis (Earth-fixed) where the secondary axis (which moves relative to the Earth as the antenna rotates) would intersect it if the axis offset $b$ were reduced to zero. The Earth-fixed position vector $\mathbf{r}_{\mathrm{b}_0}$ is multiplied by the solve-for scale factor $\alpha$, whose nominal value is unity. The vector $\mathbf{r}_{\mathrm{b}_0}$ is calculated from cylindrical or spherical station coordinates obtained from the GIN file. For cylindrical coordinates,

$$\mathbf{r}_{\mathrm{b}_0} = \begin{bmatrix} u\cos\lambda \\ u\sin\lambda \\ v \end{bmatrix} \quad \text{km} \tag{5–2}$$

where $u$ is the distance from the 1903.0 pole, $v$ is the perpendicular distance from the 1903.0 equatorial plane (positive north of the equator), and $\lambda$ is the east longitude (degrees). For spherical coordinates,

$$\mathbf{r}_{\mathrm{b}_0} = \begin{bmatrix} r\cos\phi\cos\lambda \\ r\cos\phi\sin\lambda \\ r\sin\phi \end{bmatrix} \quad \text{km} \tag{5–3}$$

where $r$ is the geocentric radius, $\phi$ is the geocentric latitude measured from the 1903.0 equatorial plane (degrees), and $\lambda$ is the east longitude. Since the Earth-fixed velocity vector $\dot{\mathbf{r}}_{\mathrm{b}}$ in term three of Eq. (5–1) acts from the user input epoch $t_0$ to the current time $t$, the station coordinates in Eqs. (5–2) and (5–3) are the values at $t_0$.

### 5.2.2 VECTOR OFFSET FROM SURVEY BENCHMARK TO TRACKING STATION

If the first term of Eq. (5–1) contains the geocentric Earth-fixed position vector of a survey benchmark, the second term is the Earth-fixed position vector from the benchmark to the station location, with rectangular components referred to the mean pole, prime meridian, and equator of 1903.0:

$$\Delta \mathbf{r}_{b_0} = d_N \mathbf{N} + d_E \mathbf{E} + d_U \mathbf{Z} \qquad \text{km} \tag{5–4}$$

where $d_N$, $d_E$, and $d_U$ are the components of this vector along the north **N**, east **E**, and zenith **Z** unit vectors at the benchmark. These unit vectors are computed from the geodetic latitude $\phi_g$ and the east longitude $\lambda$ of the benchmark:

$$\mathbf{Z} = \begin{bmatrix} \cos\phi_g \cos\lambda \\ \cos\phi_g \sin\lambda \\ \sin\phi_g \end{bmatrix} \tag{5–5}$$

$$\mathbf{N} = \begin{bmatrix} -\sin\phi_g \cos\lambda \\ -\sin\phi_g \sin\lambda \\ \cos\phi_g \end{bmatrix} \tag{5–6}$$

$$\mathbf{E} = \begin{bmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{bmatrix} \tag{5–7}$$

The geodetic latitude is computed from:

$$\phi_g = \left(\phi_g - \phi\right) + \phi \tag{5–8}$$

where $\phi$ is the geocentric latitude of the benchmark and ($\phi_g$ – $\phi$) is computed from Eq. (386) of Moyer (1971) (or an equivalent equation), which is a function of $\phi$ and the geocentric radius $r$ of the tracking station. Evaluation of Eqs. (5–5) to (5–8) requires the spherical station coordinates $r$, $\phi$, and $\lambda$ relative to the mean pole, prime meridian, and equator of 1903.0. If the input station coordinates are cylindrical, they can be converted to spherical coordinates using:

$$r = \sqrt{u^2 + v^2} \tag{5–9}$$

$$\phi = \tan^{-1}\left(\frac{v}{u}\right) \tag{5–10}$$

$$\lambda = \lambda \tag{5–11}$$

### 5.2.3 DISPLACEMENT DUE TO EARTH-FIXED VELOCITY VECTOR

The third term of Eq. (5–1) is the displacement of the tracking station due to the Earth-fixed velocity vector $\dot{\mathbf{r}}_{\mathrm{b}}$ of the tracking station (due to plate motion) acting from the user input epoch $t_0$ to the current time $t$. These epochs are measured in coordinate time ET of the Solar-System barycentric or local geocentric frame of reference. The Earth-fixed velocity vector is calculated from:

$$\dot{\mathbf{r}}_{\mathrm{b}} = \frac{1}{3.15576 \text{ x } 10^{12}} \left( v_{\mathrm{N}}\, \mathbf{N} + v_{\mathrm{E}}\, \mathbf{E} + v_{\mathrm{U}}\, \mathbf{Z} \right) \qquad \text{km/s} \tag{5–12}$$

where $v_{\mathrm{N}}$, $v_{\mathrm{E}}$, and $v_{\mathrm{U}}$ are the components of $\dot{\mathbf{r}}_{\mathrm{b}}$ along the north, east, and zenith unit vectors in cm/year. These vectors are calculated from the 1903.0 spherical coordinates of the tracking station (at the epoch $t_0$) using Eqs. (5–5) to (5–8). The same set of solve-for velocity components can be used for all tracking stations within each DSN complex.

### 5.2.4 ORIGIN OFFSET

The fourth term of Eq. (5–1) is the Earth-fixed vector $\mathbf{r}_{\mathrm{O}}$ from the center of mass of the Earth to the fixed point within the Earth, which is the origin for the input station coordinates used to compute $\mathbf{r}_{\mathrm{b}_0}$ from Eq. (5–2) or (5–3). The vector $\mathbf{r}_{\mathrm{O}}$ has rectangular components referred to the mean pole, prime meridian and equator of 1903.0:

$$\mathbf{r}_{\mathrm{O}} = \begin{bmatrix} x_{\mathrm{O}} \\ y_{\mathrm{O}} \\ z_{\mathrm{O}} \end{bmatrix} \qquad \text{km} \tag{5–13}$$

### 5.2.5 POLAR MOTION

The sum of the first four terms of Eq. (5–1) is referred to the mean pole, prime meridian, and equator of 1903.0. The fifth term of Eq. (5–1) is the polar motion correction $\Delta\mathbf{r}_{PM}$. Addition of the fifth term to the sum of the first four terms rotates this approximation to $\mathbf{r}_b$ from the mean pole, prime meridian, and equator of 1903.0 to the true pole, prime meridian, and equator of date.

In order to calculate the polar motion correction $\Delta\mathbf{r}_{PM}$, the time argument for calculating $\mathbf{r}_b$ must be converted from coordinate time ET to Coordinated Universal Time UTC, as described in Subsection 5.2.5.1. The argument UTC is used to interpolate the TP (timing and polar motion) array or the EOP (Earth Orientation Parameter) file for the $X$ and $Y$ angular coordinates of the true pole of date relative to the mean pole of 1903.0. The equation for calculating $\Delta\mathbf{r}_{PM}$ from the $X$ and $Y$ coordinates of the true pole of date is derived in Subsection 5.2.5.2.

#### 5.2.5.1 Time Transformation and Interpolation for Coordinates of the Pole

The time argument for calculating $\mathbf{r}_b$ must be converted from coordinate time ET to International Atomic Time TAI and then to Coordinated Universal Time UTC. In the Solar-System barycentric space-time frame of reference, calculate ET – TAI from the approximate expression given by Eqs. (2–26) to (2–28). In the latter equation, $t$ is the ET value of the time argument expressed in seconds past J2000.0. In the local geocentric space-time frame of reference, ET – TAI is given by Eq. (2–30). Subtract ET – TAI from ET to give TAI. Using TAI as the argument, interpolate the TP array or the EOP file for TAI – UTC and subtract it from TAI to give UTC. Using UTC as the argument, re-interpolate the TP array or the EOP file for TAI – UTC and subtract it from TAI to give a second value of UTC. Using the second value of UTC as the argument, interpolate the TP array or the EOP file for the $X$ and $Y$ angular coordinates of the true pole of date relative to the mean pole of 1903.0. Convert these coordinates from seconds of arc to radians. The $X$ and $Y$ coordinates are measured south along the 0° and 90° W meridians, respectively, of 1903.0.

#### 5.2.5.2 Polar Motion Correction

The sum of the first four terms of Eq. (5–1) is an approximation to the Earth-fixed position vector of a fixed tracking station on Earth, with rectangular components referred to the mean pole, prime meridian, and equator of 1903.0. Let this vector be denoted by:

$$\mathbf{r}_{\mathrm{b}_{1903.0}} = \begin{bmatrix} x_{\mathrm{b}} \\ y_{\mathrm{b}} \\ z_{\mathrm{b}} \end{bmatrix}_{1903.0} \qquad \mathrm{km} \qquad (5\text{–}14)$$

This vector can be rotated from the rectangular coordinate system referred to the mean pole, prime meridian, and equator of 1903.0 to the rectangular coordinate system referred to the true pole, prime meridian, and equator of date using:

$$\mathbf{r}_{\mathrm{b}_{\mathrm{true}}} = R_{\mathrm{x}}(Y)\, R_{\mathrm{y}}(X)\, \mathbf{r}_{\mathrm{b}_{1903.0}} \qquad \mathrm{km} \qquad (5\text{–}15)$$

where $R_{\mathrm{y}}(X)$ is a rotation of the Earth-fixed 1903.0 rectangular coordinate system about its y axis through the angle $X$, and $R_{\mathrm{x}}(Y)$ is a rotation of the resulting coordinate system about its x axis through the angle $Y$. The coordinate system rotation matrices for the rotation of a rectangular coordinate system about its x, y, and z axes through the angle $\theta$ (using the right-hand rule) and their derivatives with respect to $\theta$ are given by:

$$R_{\mathrm{x}}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \qquad \frac{dR_{\mathrm{x}}(\theta)}{d\theta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta & \cos\theta \\ 0 & -\cos\theta & -\sin\theta \end{bmatrix} \qquad (5\text{–}16)$$

$$R_{\mathrm{y}}(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \qquad \frac{dR_{\mathrm{y}}(\theta)}{d\theta} = \begin{bmatrix} -\sin\theta & 0 & -\cos\theta \\ 0 & 0 & 0 \\ \cos\theta & 0 & -\sin\theta \end{bmatrix} \qquad (5\text{–}17)$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \frac{dR_z(\theta)}{d\theta} = \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (5\text{–}18)$$

The polar motion correction $\Delta\mathbf{r}_{\mathrm{PM}}$ in Eq. (5–1) is defined to be:

$$\Delta\mathbf{r}_{\mathrm{PM}} = \mathbf{r}_{\mathrm{b}_{\mathrm{true}}} - \mathbf{r}_{\mathrm{b}_{1903.0}} \qquad \mathrm{km} \qquad (5\text{–}19)$$

Substituting Eq. (5–15) gives:

$$\Delta\mathbf{r}_{\mathrm{PM}} = \left[R_x(Y)\,R_y(X) - I\right]\mathbf{r}_{\mathrm{b}_{1903.0}} \qquad \mathrm{km} \qquad (5\text{–}20)$$

where $I$ is the 3 x 3 identity matrix:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (5\text{–}21)$$

Eq. (5–20) is evaluated by substituting Eqs. (5–14), (5–16), (5–17), and (5–21). The two coordinate system rotation matrices are evaluated using the first-order approximations: $\cos X = \cos Y = 1$, $\sin X = X$, and $\sin Y = Y$. In the product of the two matrices, the second-order term $XY$ is ignored. The resulting expression for the polar motion correction is:

$$\Delta\mathbf{r}_{\mathrm{PM}} = \begin{bmatrix} -z_{\mathrm{b}}X \\ z_{\mathrm{b}}Y \\ x_{\mathrm{b}}X - y_{\mathrm{b}}Y \end{bmatrix} \qquad \mathrm{km} \qquad (5\text{–}22)$$

where, from Eq. (5–14), $x_{\mathrm{b}}$, $y_{\mathrm{b}}$, and $z_{\mathrm{b}}$ are rectangular components referred to the mean pole, prime meridian, and equator of 1903.0 of the Earth-fixed position vector of a fixed tracking station on Earth, calculated from the first four terms of Eq. (5–1).

The effect of the neglected second-order terms in Eq. (5–22) on the Earth-fixed position vector of a tracking station is less than 0.1 mm. The components of the polar motion correction (5–22) are less than 20 m.

### 5.2.6 SOLID EARTH TIDES

The sixth term of Eq. (5–1) is the displacement $\Delta\mathbf{r}_{\mathrm{SET}}$ of a fixed tracking station on Earth due to solid Earth tides. The Earth-fixed rectangular components of this vector are referred to the true pole, prime meridian, and equator of date. Subsection 5.2.6.1 gives the expression for the tidal potential $W_2$ at the tracking station, which is calculated from the Earth-fixed position vectors of the tracking station, the Moon, and the Sun. Subsection 5.2.6.2 derives the equations for the first-order displacement of the tracking station due to solid earth tides. The components of this displacement are calculated from $W_2$ and its derivatives with respect to the tracking station coordinates. Subsection 5.2.6.3 expresses the tidal potential as a spherical harmonic expansion. The equations for calculating the angular argument for each term (a specific tide) of the tidal potential are given in that section and in Subsection 5.2.6.4. The displacement of the tracking station due to each term of the tidal potential is proportional to the Love number $h_2$ in the radial direction and the Love number $l_2$ in the north and east directions. These Love numbers are frequency dependent and are different for each term of the tide-generating potential. However, the equation in Subsection 5.2.6.2 for the first-order tidal displacement uses constant values of $h_2$ and $l_2$. Subsection 5.2.6.5 gives a second-order correction to the tidal displacement of a tracking station. It is a correction to the radial displacement due to the departure of the value of $h_2$ for a particular term of the astronomical tide-generating potential (the so-called $K_1$ diurnal tide) from the constant value of $h_2$ used in calculating the first-order tidal displacement. Subsection 5.2.6.6 develops expressions for the constant part of the displacement of a tracking station due to solid Earth tides. This permanent tidal displacement is included in the expression for the first-order displacement. If the permanent tidal displacement was subtracted from the sum of the first-order and second-order tidal displacements, then the estimated coordinates of the tracking station would include the permanent tidal displacement. However, this is not done by international agreement.

### 5.2.6.1 Tidal Potential $W_2$

The tidal potential can be represented to sufficient accuracy by the spherical harmonic function $W_2$, which is of the second degree. Second-degree tidal displacements are on the order of 50 cm. Third-degree tidal displacements are less than a centimeter and are ignored. The tidal potential $W_2$, which is based upon a spherical Earth and a point-mass perturbing Moon or Sun, is given by Eq. (1.11) on p. 15 of Melchior (1966). Adding the terms due to the Moon and the Sun gives:

$$W_2 = \sum_{j=2}^{3} \frac{\mu_j r^2}{2 R_j^3} \left( 3 \cos^2 z_j - 1 \right) \qquad \mathrm{km}^2/\mathrm{s}^2 \qquad (5\text{–}23)$$

where

$j$ = disturbing body (2 = Moon, 3 = Sun).
$\mu_j$ = gravitational constant of body $j$, $\mathrm{km}^3/\mathrm{s}^2$.
$R_j$ = geocentric radial coordinate of body $j$, km.
$r$ = geocentric radial coordinate of tracking station ($W_2$ is the tidal potential at that point), km.
$z_j$ = angle measured at the center of the Earth from the tracking station to body $j$.

In order to calculate $\cos z_j$, let

$\mathbf{R}_j$ = geocentric Earth-fixed position vector of body $j$, with rectangular components referred to the true pole, prime meridian, and equator of date.
$\mathbf{r}$ = geocentric Earth-fixed position vector of the tracking station, with rectangular components referred to the true pole, prime meridian, and equator of date.

The unit vectors $\hat{\mathbf{R}}_j$ and $\hat{\mathbf{r}}$ are given by:

$$\hat{\mathbf{R}}_j = \frac{\mathbf{R}_j}{R_j} \tag{5–24}$$

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} \tag{5–25}$$

where $R_j$ and $r$ are the magnitudes of $\mathbf{R}_j$ and $\mathbf{r}$, respectively. Then,

$$\cos z_j = \hat{\mathbf{r}} \cdot \hat{\mathbf{R}}_j \tag{5–26}$$

Melchior (1966) calculated the rectangular components of the acceleration at a tracking station on Earth due to the disturbing body (the Moon or the Sun) minus the corresponding acceleration components at the center of the Earth. He used these relative acceleration components to calculate the variation *dg* in the radial gravity *g* (on a spherical Earth) and the deflection *e* of the vertical due to disturbing body *j*. His expression for *dg* is his Eq. (1.10):

$$dg = -\mu_j \frac{r}{R_j^{\,3}}\left(3\cos^2 z_j - 1\right) = g\left(\frac{\mu_j}{\mu_{\mathrm{E}}}\right)\left(\frac{r}{R_j}\right)^3 \left(1 - 3\cos^2 z_j\right) \quad \mathrm{km/s^2} \tag{5–27}$$

where $g$ is the acceleration of gravity at the tracking station given by:

$$g = \frac{\mu_{\mathrm{E}}}{r^2} \qquad \mathrm{km/s^2} \tag{5–28}$$

where

$\mu_{\mathrm{E}}$ = gravitational constant of the Earth, $\mathrm{km^3/s^2}$.

Eq. (5–27) can be obtained from the term of Eq. (5–23) for disturbing body *j* using:

$$dg = -\frac{\partial W_2}{\partial r} \tag{5–29}$$

Melchior’s expression for the deflection $e$ of the vertical is his Eq. (1.9):

$$e = \frac{3}{2}\left(\frac{\mu_j}{\mu_{\mathrm{E}}}\right)\left(\frac{r}{R_j}\right)^3 \sin 2z_j \tag{5–30}$$

This equation can be obtained from the term of Eq. (5–23) for disturbing body $j$ using:

$$e = -\frac{1}{gr}\frac{\partial W_2}{\partial z_j} \tag{5–31}$$

### 5.2.6.2 First-Order Displacement of the Tracking Station Due to Solid Earth Tides

From Melchior (1966), p. 114, Eq. (2.19), the components of the displacement of the tracking station due to solid Earth tides are given by the following functions of the tidal potential $W_2$ and its partial derivatives with respect to the geocentric latitude $\phi$ and longitude $\lambda$ of the tracking station:

$$s_r = \frac{h_2}{g} W_2 \qquad \text{km} \tag{5–32}$$

$$s_\phi = \frac{l_2}{g}\frac{\partial W_2}{\partial \phi} \qquad \text{km} \tag{5–33}$$

$$s_\lambda = \frac{l_2}{g\cos\phi}\frac{\partial W_2}{\partial \lambda} \qquad \text{km} \tag{5–34}$$

where the displacement $s_r$ is in the geocentric radial direction. The transverse displacements $s_\phi$ and $s_\lambda$ are normal to the geocentric radius, directed toward the north and east, respectively. The acceleration of gravity $g$ at the tracking station is given by Eq. (5–28). The quantities $h_2$ and $l_2$ are second-degree Love numbers. From International Earth Rotation Service (1992), p. 57, the nominal values of these Love numbers are:

$$\begin{aligned} h_2 &= 0.6090 \\ l_2 &= 0.0852 \end{aligned} \tag{5–35}$$

Wahr (1981), p. 699, Table 5 lists these numerical values as the appropriate values for any semi-diurnal tide component.

Eq. (5–32) follows because the geoid (mean sea level) is an equipotential surface, where the potential is the sum of the gravitational and centrifugal potential (see Subsection 5.2.8). Addition of the tidal potential $W_2$ requires the radial displacement of the ocean given by Eq. (5–32) with $h_2$ = unity in order to keep the potential constant. Eqs. (5–33) and (5–34) with $l_2$ = unity give the transverse displacements of the ocean. If these equations are multiplied by $g$ and divided by $r$, the right-hand sides give the transverse tidal accelerations, which are balanced by the left-hand sides, which are the components of gravity at the displaced positions normal to the geocentric radial at the original position. These accelerations are equal and opposite.

The displacement of the Earth-fixed position vector $\mathbf{r}_\mathrm{b}$ of the tracking station due to solid Earth tides is given by:

$$\Delta \mathbf{r}_\mathrm{b} = s_r\, \hat{\mathbf{r}} + s_\phi\, \mathbf{N} + s_\lambda\, \mathbf{E} \qquad \text{km} \tag{5–36}$$

where, for a spherical Earth, the north and east unit vectors are given by:

$$\mathbf{N} = \begin{bmatrix} -\sin\phi \cos\lambda \\ -\sin\phi \sin\lambda \\ \cos\phi \end{bmatrix} \tag{5–37}$$

$$\mathbf{E} = \begin{bmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{bmatrix} \tag{5–38}$$

The unit vector $\hat{\mathbf{r}}$ in the geocentric radial direction is given by:

$$\hat{\mathbf{r}} = \begin{bmatrix} \cos\phi\cos\lambda \\ \cos\phi\sin\lambda \\ \sin\phi \end{bmatrix} \tag{5–39}$$

and

$$\frac{\partial\hat{\mathbf{r}}}{\partial\phi} = \mathbf{N} \tag{5–40}$$

$$\frac{\partial\hat{\mathbf{r}}}{\partial\lambda} = (\cos\phi)\,\mathbf{E} \tag{5–41}$$

The geocentric latitude $\phi$, longitude $\lambda$, all Earth-fixed vectors and unit vectors appearing in this section, and the displacement $\Delta\mathbf{r}_\mathrm{b}$ are referred to the Earth-fixed rectangular coordinate system aligned with the true pole, prime meridian, and equator of date.

Evaluating $s_r$ using Eqs. (5–32), (5–28), (5–23), and (5–26) gives:

$$s_r = h_2 \sum_{j=2}^{3} \frac{\mu_j}{\mu_\mathrm{E}} \frac{r^4}{R_j{}^3} \left[ \frac{3}{2} \left( \hat{\mathbf{R}}_j \cdot \hat{\mathbf{r}} \right)^2 - \frac{1}{2} \right] \qquad \text{km} \tag{5–42}$$

Evaluating $s_\phi$ using Eqs. (5–33), (5–28), (5–23), (5–26), and (5–40) gives:

$$s_\phi = 3 l_2 \sum_{j=2}^{3} \frac{\mu_j}{\mu_\mathrm{E}} \frac{r^4}{R_j{}^3} \left( \hat{\mathbf{R}}_j \cdot \hat{\mathbf{r}} \right) \left( \hat{\mathbf{R}}_j \cdot \mathbf{N} \right) \qquad \text{km} \tag{5–43}$$

Evaluating $s_\lambda$ using Eqs. (5–34), (5–28), (5–23), (5–26), and (5–41) gives:

$$s_\lambda = 3l_2 \sum_{j=2}^{3} \frac{\mu_j}{\mu_\mathrm{E}} \frac{r^4}{R_j{}^3} \left( \hat{\mathbf{R}}_j \cdot \hat{\mathbf{r}} \right) \left( \hat{\mathbf{R}}_j \cdot \mathbf{E} \right) \qquad \mathrm{km} \qquad (5\text{–}44)$$

After substituting Eqs. (5–42) to (5–44) into (5–36), the sum of terms two and three of (5–36) is given by a common factor multiplied by the following function, which can be expressed as:

$$\left( \hat{\mathbf{R}}_j \cdot \mathbf{N} \right) \mathbf{N} + \left( \hat{\mathbf{R}}_j \cdot \mathbf{E} \right) \mathbf{E} = \hat{\mathbf{R}}_j - \left( \hat{\mathbf{R}}_j \cdot \hat{\mathbf{r}} \right) \hat{\mathbf{r}} \qquad (5\text{–}45)$$

Hence, substituting Eqs. (5–42) to (5–44) into (5–36) and then substituting Eq. (5–45) into the resulting expression gives the following equation for the first-order term of the displacement of the Earth-fixed tracking station due to solid Earth tides:

$$\Delta \mathbf{r}_\mathrm{b} = \sum_{j=2}^{3} \frac{\mu_j}{\mu_\mathrm{E}} \frac{r^4}{R_j{}^3} \left\{ 3l_2 \left( \hat{\mathbf{R}}_j \cdot \hat{\mathbf{r}} \right) \hat{\mathbf{R}}_j + \left[ 3 \left( \frac{h_2}{2} - l_2 \right) \left( \hat{\mathbf{R}}_j \cdot \hat{\mathbf{r}} \right)^2 - \frac{h_2}{2} \right] \hat{\mathbf{r}} \right\}$$
$$\mathrm{km} \qquad (5\text{–}46)$$

This is Eq. (6) on p. 57 of International Earth Rotation Service (1992).

Eq. (5–46) was derived assuming that the solid Earth responds instantaneously to the tide-producing potential $W_2$. In order to allow for a delay in the elastic response of the solid Earth to $W_2$, the radial, north, and east components of the displacement of the tracking station will be computed from Eqs. (5–42) to (5–44) using phase-shifted values of the unit vectors $\hat{\mathbf{r}}$, **N**, and **E**:

$$\hat{\mathbf{r}}_\mathrm{p} = L\,\hat{\mathbf{r}} \qquad \hat{\mathbf{r}} \rightarrow \mathbf{N},\ \mathbf{E} \qquad (5\text{–}47)$$

where $L$ is a positive rotation of the Earth-fixed rectangular coordinate system about its z axis through the angle $\psi$ (see Eq. 5–18):

$$L = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5–48}$$

The phase shift $\psi$ will be an input constant (nominally 0°). If Eqs. (5–37) to (5–39) and (5–48) are substituted into Eq. (5–47), it is seen that the phase-shifted unit vectors $\mathbf{N}_\mathrm{p}$, $\mathbf{E}_\mathrm{p}$, and $\hat{\mathbf{r}}_\mathrm{p}$ can be calculated from Eqs. (5–37) to (5–39) with the longitude $\lambda$ of the tracking station replaced with $\lambda - \psi$. Using these phase-shifted unit vectors to calculate the radial, north, and east components of the tidal displacement of the tracking station from Eqs. (5–42) to (5–44) causes the peak radial tide to occur $\psi/\omega_\mathrm{E}$ seconds after the tracking station meridian passes under the disturbing body (the Moon or the Sun), where $\omega_\mathrm{E}$ is the angular rotation rate of the Earth.

The radial, north, and east displacements calculated from Eqs. (5–42) to (5–44) using the phase-shifted unit vectors $\mathbf{N}_\mathrm{p}$, $\mathbf{E}_\mathrm{p}$, and $\hat{\mathbf{r}}_\mathrm{p}$ are substituted into Eq. (5–36). However, the unit vectors $\mathbf{N}$, $\mathbf{E}$, and $\hat{\mathbf{r}}$ appearing explicitly in Eq. (5–36) are not phase shifted. Before substituting Eq. (5–45) into this phase-shifted version of Eq. (5–36), two modifications must be made. First, evaluate Eq. (5–45) with the phase-shifted unit vectors $\mathbf{N}_\mathrm{p}$, $\mathbf{E}_\mathrm{p}$, and $\hat{\mathbf{r}}_\mathrm{p}$:

$$\left(\hat{\mathbf{R}}_j \cdot \mathbf{N}_\mathrm{p}\right)\mathbf{N}_\mathrm{p} + \left(\hat{\mathbf{R}}_j \cdot \mathbf{E}_\mathrm{p}\right)\mathbf{E}_\mathrm{p} = \hat{\mathbf{R}}_j - \left(\hat{\mathbf{R}}_j \cdot \hat{\mathbf{r}}_\mathrm{p}\right)\hat{\mathbf{r}}_\mathrm{p} \tag{5–49}$$

Next, pre-multiply each term of this equation by $L^\mathrm{T}$, which gives:

$$\left(\hat{\mathbf{R}}_j \cdot \mathbf{N}_\mathrm{p}\right)\mathbf{N} + \left(\hat{\mathbf{R}}_j \cdot \mathbf{E}_\mathrm{p}\right)\mathbf{E} = L^\mathrm{T}\hat{\mathbf{R}}_j - \left(\hat{\mathbf{R}}_j \cdot \hat{\mathbf{r}}_\mathrm{p}\right)\hat{\mathbf{r}} \tag{5–50}$$

Substituting Eq. (5–50) into the phase-shifted version of Eq. (5–36) gives the phase-shifted version of Eq. (5–46):

$$\Delta\mathbf{r}_\mathrm{b} = \sum_{j=2}^{3} \frac{\mu_j}{\mu_\mathrm{E}} \frac{r^4}{R_j^{\,3}} \left\{ 3l_2\left(\hat{\mathbf{R}}_j \cdot \hat{\mathbf{r}}_\mathrm{p}\right) L^\mathrm{T}\hat{\mathbf{R}}_j + \left[ 3\left(\frac{h_2}{2} - l_2\right)\left(\hat{\mathbf{R}}_j \cdot \hat{\mathbf{r}}_\mathrm{p}\right)^2 - \frac{h_2}{2} \right]\hat{\mathbf{r}} \right\} \quad \text{km} \tag{5–51}$$

If the phase shift $\psi$ is set to zero, this equation reduces to Eq. (5–46). Eq. (5–51) is the final expression for the first-order term of the displacement of the Earth-fixed tracking station due to solid Earth tides.

Eq. (5–51) is evaluated by executing the following steps:

1. The geocentric Earth-fixed position vector $\mathbf{r}$ of the tracking station, with rectangular components referred to the true pole, prime meridian, and equator of date is given by the sum of the first five terms of Eq. (5–1). Calculate the magnitude $r$ of the vector $\mathbf{r}$, and then calculate the unit vector $\hat{\mathbf{r}}$ to the tracking station from Eq. (5–25). Using the input phase shift $\psi$, calculate $L$ from Eq. (5–48) and the phase-shifted unit vector $\hat{\mathbf{r}}_{\mathrm{p}}$ to the tracking station from Eq. (5–47). In evaluating Eq. (5–51), the unit vector $\hat{\mathbf{r}}$ is used once and the phase-shifted unit vector $\hat{\mathbf{r}}_{\mathrm{p}}$ is used twice.

2. The time argument for calculating the geocentric Earth-fixed and space-fixed position vectors of the fixed tracking station on Earth is coordinate time ET in the Solar-System barycentric or local geocentric space-time frame of reference. Using this ET time argument, interpolate the planetary ephemeris for the geocentric (E) space-fixed position vectors of the Moon (M) and the Sun (S):

$$\mathbf{r}_{\mathrm{M}}^{\mathrm{E}} \quad , \quad \mathbf{r}_{\mathrm{S}}^{\mathrm{E}}$$

3. Using the ET time argument, calculate the 3 x 3 Earth-fixed to space-fixed transformation matrix $T_{\mathrm{E}}$ (using the formulation given in Section 5.3).

4. Transform the geocentric space-fixed position vectors of the Moon and the Sun to the corresponding Earth-fixed position vectors, with rectangular components referred to the true pole, prime meridian, and equator of date:

$$\mathbf{R}_2 = {T_{\mathrm{E}}}^{\mathrm{T}}\, \mathbf{r}_{\mathrm{M}}^{\mathrm{E}} \qquad \text{km} \qquad (5\text{–}52)$$

$$\mathbf{R}_3 = T_{\mathrm{E}}{}^{\mathrm{T}}\ \mathbf{r}_{\mathrm{S}}^{\mathrm{E}} \qquad \text{km} \qquad (5\text{–}53)$$

where the superscript T indicates the transpose of the matrix. Calculate the magnitudes $R_2$ and $R_3$ of these vectors. Then calculate the unit vector $\hat{\mathbf{R}}_2$ to the Moon and the unit vector $\hat{\mathbf{R}}_3$ to the Sun from Eq. (5–24).

5. Using $r$, $\hat{\mathbf{r}}$, $\hat{\mathbf{r}}_{\mathrm{p}}$, and $L$ from step 1; $R_2$, $R_3$, $\hat{\mathbf{R}}_2$, and $\hat{\mathbf{R}}_3$ from step 4; the input values of the Love numbers $h_2$ and $l_2$; and the gravitational constants $\mu_2$ of the Moon, $\mu_3$ of the Sun, and $\mu_{\mathrm{E}}$ of the Earth obtained from the planetary ephemeris, calculate the first-order term of the Earth-fixed displacement $\Delta\mathbf{r}_{\mathrm{SET}}$ (term six of Eq. 5–1) of the tracking station due to solid Earth tides from Eq. (5–51).

#### 5.2.6.3 Expansion of the Tidal Potential

Cartwright and Tayler (1971) and Wahr (1981) express the tidal potential $W$ (divided by the acceleration of gravity $g$ given by Eq. 5–28) as a spherical harmonic expansion with time-dependent (*i.e.,* sinusoidal) coefficients[1]. However, their equations are vague and ambiguous. These equations were compared to the corresponding equations in Melchior (1966). This comparison enabled the exact form of the spherical harmonic expansion of $W/g$ to be determined. It is given by:

$$\frac{W}{g} = \sum_{n=2}^{3} \sum_{m=0}^{n} \sum_{s} H_s W_n^m(\phi) \begin{matrix}\cos\\ \sin\end{matrix}(\theta_s + m\lambda) \qquad \text{m} \qquad (5\text{–}54)$$

where the cosine applies when $(n + m)$ is even and the sine applies when $(n + m)$ is odd. Let $W_n^m(\phi,\lambda)$ be the normalized spherical harmonic of degree $n$ and order $m$ in the geocentric latitude $\phi$ and longitude $\lambda$ of the point on a spherical

[1] Since terms of different degree are included, the subscript 2 of $W_2$ (indicating degree 2) is dropped.

Earth where $W/g$ is evaluated. From Eq. (10) of Cartwright and Tayler (1971) or Eq. (2.4) of Wahr (1981), it is given by:

$$W_n^m(\phi,\lambda) = (-1)^m \left[\frac{2n+1}{4\pi} \cdot \frac{(n-m)!}{(n+m)!}\right]^{\frac{1}{2}} P_n^m(\sin\phi)\, e^{im\lambda} \tag{5–55}$$

where $P_n^m(\sin\phi)$ is the associated Legendre function of sine latitude. From Eq. (3.49) of Jackson (1975), without the factor $(-1)^m$ which is included separately in Eq. (5–55),

$$P_n^m(\sin\phi) = \cos^m\phi \, \frac{d^m}{d(\sin\phi)^m} P_n(\sin\phi) \tag{5–56}$$

which is Eq. (155) of Moyer (1971). In Eq. (5–56), $P_n(\sin\phi)$ is the Legendre polynomial of degree $n$ in sin $\phi$. From Eq. (3.16) of Jackson (1975),

$$P_n(\sin\phi) = \frac{1}{2^n n!} \frac{d^n}{d(\sin\phi)^n} \left(\sin^2\phi - 1\right)^n \tag{5–57}$$

The Legendre polynomials can be computed from this equation or can be computed recursively from Eqs. (175) to (177) of Moyer (1971). Substituting Eq. (5–57) into Eq. (5–56) gives $P_n^m(\sin\phi)$ as a direct function of sin $\phi$:

$$P_n^m(\sin\phi) = \frac{\cos^m\phi}{2^n n!} \frac{d^{n+m}}{d(\sin\phi)^{n+m}} \left(\sin^2\phi - 1\right)^n \tag{5–58}$$

This is Eq. (11) of Cartwright and Tayler (1971) and Eq. (2.5) of Wahr (1981). In Eq. (5–54), $W_n^m(\phi)$ is $W_n^m(\phi,\lambda)$ given by Eq. (5–55) without the factor $e^{im\lambda}$. That is,

$$W_n^m(\phi) = e^{-im\lambda} W_n^m(\phi,\lambda) \tag{5–59}$$

which is a function of the geocentric latitude $\phi$. The function $W_n^m(\phi)$ given by Eqs. (5–55) to (5–59) has been evaluated for $n$ = 2 and 3 for $m$ = 0 to $n$ in Table 2 of Cartwright and Tayler (1971) and on pages 99 and 100 of Jackson (1975). However, these functions are expressed in terms of sines and cosines of the co-latitude (90° – $\phi$).

Each term of Eq. (5–54) corresponds to a specific solid Earth tide. The summation is over the degree $n$, the order $m$ which varies from 0 to $n$, and all of the tides $s$ for a given degree $n$ and order $m$. For each tide $s$, $H_s$ is the amplitude (in meters) and $\theta_s$ is the phase angle or astronomical argument, which is defined by the sequence of six integers $n_1$ through $n_6$. Given these integers, the value of $\theta_s$ at a given time $t$ is computed from the equation on p. 53 of International Earth Rotation Service (1992):

$$\theta_s = \sum_{i=1}^{6} n_i \beta_i \tag{5–60}$$

where $\beta_1$ through $\beta_6$ are the Doodson variables. They are astronomical angles which are computed from sums and differences of the five fundamental angular arguments of the nutation series and mean sidereal time. The definitions of $\beta_1$ through $\beta_6$ and the polynomials for computing them as a function of time are given in Subsection 5.2.6.4. For each tide, the six integers $n_1$ through $n_6$ are coded into the Doodson argument number (see p. 65 of International Earth Rotation Service (1992)):

$$n_1(n_2+5)(n_3+5).(n_4+5)(n_5+5)(n_6+5) \tag{5–61}$$

This is a sequence of six positive integers separated by a central dot. The Doodson variables $\beta_2$ through $\beta_6$ are slowly varying angles. However, $\beta_1$ contains mean sidereal time and has a frequency of about 1 cycle/day. Also, the integer $n_1$ in the Doodson argument number for each tide is equal to the order $m$:

$$n_1 = m \tag{5–62}$$

Hence, from Eq. (5–60), the frequency of $\theta_s$ in Eq. (5–54) is about 1 cycle/day for all diurnal tides ($n_1 = m = 1$) and about 2 cycles/day for all semi-diurnal tides ($n_1 = m = 2$). For all long-period tides, $n_1 = m = 0$. Since $\theta_s$ contains the term $n_1\beta_1 = m\beta_1$ which contains the term $m\theta_M$, where $\theta_M$ is mean sidereal time, the argument $\theta_s + m\lambda$ in Eq. (5–54) contains the term $m(\theta_M + \lambda)$.

Cartwright and Tayler (1971) gives values of the amplitude $H_s$ (in meters) and the Doodson argument number for a large number of tides. This information for tides of the second degree ($n = 2$) is given in Tables 4a, b, and c. These tables apply for long-period tides ($m = 0$), diurnal tides ($m = 1$), and semi-diurnal tides ($m = 2$), respectively. The same information for tides of the third degree ($n = 3$) is given in Tables 5a, b, and c. Table 5d applies for ter-diurnal tides ($m = 3$) of the third degree. For each tide, column 1 lists the six integers $n_1$ through $n_6$. Columns 2, 3, and 4 give the amplitude $H_s$ for three different time periods, which are identified in Table 3 of this reference. We will use the values from the latest time period (May 23, 1951 to May 23, 1969), which are given in column 4. Column 5 gives the six integers $n_1$ through $n_6$ coded into the Doodson argument number. We do not use the last two columns of these tables. After correcting a small error, the information for the second-degree tides in Tables 4a, b, and c of Cartwright and Tayler (1971) was recalculated and presented in Tables 1a, b, and c of Cartwright and Edden (1973). The information given for the third-degree tides in Tables 5a, b, c, and d of Cartwright and Tayler (1971) was unaffected by the small error. From Cartwright and Tayler (1971), lunar tides were computed for degree 2 and 3, and solar tides were computed for degree 2 only. From the above-mentioned tables, the amplitude $H_s$ of individual second-degree tides is up to about 0.63 meters (for the semi-diurnal lunar tide $M_2$, Doodson argument 255.555). The third-degree tides have amplitudes $H_s$ up to about 0.008 meters.

#### 5.2.6.4 The Doodson Variables

In Eq. (5–54), $\theta_s$ is the astronomical argument for a particular tide $s$. The argument $\theta_s$ is defined by the sequence of six integers $n_1$ through $n_6$ (which are coded into the Doodson argument number) and is calculated from Eq. (5–60). In this equation, $\beta_1$ through $\beta_6$ are the Doodson variables. This section defines them and gives equations for computing them.

From pages 53 and 54 of International Earth Rotation Service (1992), the six Doodson variables $\beta_1$ through $\beta_6$ are functions of the five fundamental arguments $l$, $l'$, $F$, $D$, and $\Omega$ (defined below) of the nutation series and mean sidereal time $\theta_M$:

$$
\begin{aligned}
\beta_2 &= s && = F + \Omega && = \text{Mean Longitude of the Moon} \\
\beta_3 &= h && = s - D && = \text{Mean Longitude of the Sun} \\
\beta_4 &= p && = s - l && = \text{Longitude of the Moon's Mean Perigee} \\
\beta_5 &= N' && = -\Omega && = \text{Negative of the Longitude of the} \\
& && && \quad \text{Moon's Mean Ascending Node} \\
\beta_6 &= p_1 && = s - D - l' && = \text{Longitude of the Sun's Mean Perigee} \\
\beta_1 &= \tau && = \theta_M + \pi - s && = \text{Mean Lunar Time (Greenwich Hour} \\
& && && \quad \text{Angle of Mean Moon plus 12 hours)}
\end{aligned}
\tag{5–63}
$$

From p. 32 of International Earth Rotation Service (1992), or p. 98 of Seidelman (1982), the fundamental arguments of the nutation series are:

$$
\begin{aligned}
l &= \text{Mean Anomaly of the Moon} \\
&= 134^\circ 57' 46''.733 + \left(1325^{\mathrm{r}} + 198^\circ 52' 02''.633\right) T + 31''.310\, T^2 + 0''.064\, T^3 \\
l' &= \text{Mean Anomaly of the Sun} \\
&= 357^\circ 31' 39''.804 + \left(99^{\mathrm{r}} + 359^\circ 03' 01''.224\right) T - 0''.577\, T^2 - 0''.012\, T^3 \\
F &= \text{Mean Argument of Latitude of the Moon} \\
&= L - \Omega,\ \text{where } L = \text{Mean Longitude of the Moon and } \Omega \text{ is defined below} \\
&= 93^\circ 16' 18''.877 + \left(1342^{\mathrm{r}} + 82^\circ 01' 03''.137\right) T - 13''.257\, T^2 + 0''.011 T^3 \\
D &= \text{Mean Elongation of the Moon from the Sun} \\
&= L - L_{\mathrm{s}},\ \text{where } L_{\mathrm{s}} = \text{Mean Longitude of the Sun} \\
&= 297^\circ 51' 01''.307 + \left(1236^{\mathrm{r}} + 307^\circ 06' 41''.328\right) T - 6''.891 T^2 + 0''.019\, T^3 \\
\Omega &= \text{Longitude of the Mean Ascending Node of the Lunar Orbit on the} \\
&\qquad \text{Ecliptic, Measured from the Mean Equinox of Date} \\
&= 125^\circ 02' 40''.280 - \left(5^{\mathrm{r}} + 134^\circ 08' 10''.539\right) T + 7''.455\, T^2 + 0''.008\, T^3
\end{aligned}
\tag{5–64}
$$

where $1^{\mathrm{r}} = 360^\circ = 1296000''$ and

$$
\begin{aligned}
T &= \text{Julian centuries of 36525 days of 86400 s of coordinate time ET} \\
&\quad \text{(in the Solar - System barycentric or local geocentric frame of reference)} \\
&\quad \text{past January 1, 2000, } 12^{\mathrm{h}}\ \text{ET (J2000.0; JED 245,1545.0)} \\
&= \frac{\mathrm{ET}}{86400 \times 36525}
\end{aligned}
\tag{5–65}
$$

where

$$\mathrm{ET} \quad = \quad \text{seconds of coordinate time past J2000.0}$$

Converting Eqs. (5–64) to arcseconds gives

$$
\begin{aligned}
l &= 485{,}866''.733 + 1{,}717{,}915{,}922''.633\,T + 31''.310\,T^2 + 0''.064\,T^3 \\
l' &= 1{,}287{,}099''.804 + 129{,}596{,}581''.224\,T - 0''.577\,T^2 - 0''.012\,T^3 \\
F &= 335{,}778''.877 + 1{,}739{,}527{,}263''.137\,T - 13''.257\,T^2 + 0''.011\,T^3 \\
D &= 1{,}072{,}261''.307 + 1{,}602{,}961{,}601''.328\,T - 6''.891\,T^2 + 0''.019\,T^3 \\
\Omega &= 450{,}160''.280 - 6{,}962{,}890''.539\,T + 7''.455\,T^2 + 0''.008\,T^3
\end{aligned}
\tag{5–66}
$$

Calculation of the Doodson variable $\beta_1$ requires mean sidereal time $\theta_M$. The ODP code calculates true sidereal time $\theta$, which is $\theta_M$ plus a nutation term, which is less than $10^{-4}$ rad. From Eq. (5–42), the radial solid Earth tide varies from about +32 cm to −16 cm. If the maximum positive displacement were calculated from Eqs. (5–32) and (5–54) (instead of Eq. 5–51) using true sidereal time $\theta$ instead of mean sidereal time $\theta_M$ to calculate $\beta_1$, which is used to calculate $\theta_s$ from Eq. (5–60), the error would be less than 0.06 mm. However, we only use the expansion of the tidal potential and the Doodson variables to calculate the second-order correction to the tidal displacement of the tracking station (Section 5.2.6.5) and the tracking station displacement due to ocean loading (Section 5.2.7). These corrections are no more than a few centimeters and the error in computing them from $\theta$ instead of $\theta_M$ is less than 0.002 mm, which is negligible. Hence, $\beta_1$ in Eq. (5–63) is calculated from $\theta$ instead of $\theta_M$.

The formulation for calculating sidereal time $\theta$ is given in Section 5.3.6.2. This formulation includes the transformation of the time argument from coordinate time ET to Universal Time UT1.

Calculation of the six Doodson variables $\beta_1$ through $\beta_6$ from Eqs. (5–63) requires the calculation of $l$, $l'$, $F$, $D$, and $\Omega$ from Eqs. (5–66), where $T$ is computed from the ET value of the epoch using Eq. (5–65). These five angles must be converted from arcseconds to radians by dividing by 206,264.806,247,096. The ET value of the epoch is also used to calculate true sidereal time $\theta$, which is used instead of mean sidereal time $\theta_M$ in calculating $\beta_1$.

### 5.2.6.5 Second-Order Correction to the Tidal Displacement of the Tracking Station

Second-order tidal displacements account for the departure of the frequency-dependent Love numbers $h_2$ and $l_2$ from the constant values (Eq. 5–35) used to calculate the first-order tidal displacement from Eq. (5–51).

The tidal displacements in the radial, north, and east directions could be computed from Eqs. (5–32) to (5–34), where $W_2/g$ is replaced by $W/g$ given by Eq. (5–54). In these equations, $h_2$ and $l_2$ are frequency dependent. That is, they are different for each term of Eq. (5–54) that they multiply. The second-order tidal displacements can be computed from Eqs. (5–32) to (5–34) and (5–54) by replacing $h_2$ and $l_2$ with $\Delta h_2$ and $\Delta l_2$, which are the departures of $h_2$ and $l_2$ (for a particular tide or term of Eq. 5–54) from the constant values (Eq. 5–35) used in computing the first-order tidal displacement from Eq. (5–51).

The number of terms contained in the second-order tidal displacement depends upon the error criterion used. International Earth Rotation Service (1992), p. 57, used a cutoff of 5 mm (which I adopt) and obtained one term in the radial direction and no terms in the north and east directions.

The frequency-dependent values of $h_2$ and $l_2$ are given in Table 5 on p. 699 of Wahr (1981). There are significant variations of $h_2$ and $l_2$ (denoted as $h_0$ and $l_0$ by Wahr) with the frequency of the individual diurnal ($n$ = 2, $m$ = 1) tides. The values given by Eq. (5–35) apply for all semi-diurnal ($n$ = 2, $m$ = 2) tides. Hence, there are no second-order corrections for the semi-diurnal tides. Constant values of $h_2$ and $l_2$ (which differ from those in Eq. 5–35) apply for all long-period ($n$ = 2, $m$ = 0) tides.

The second-order tidal displacements in the north and east directions are a maximum of about 1 mm each, which can be ignored. The only tide that produces a radial second-order displacement greater than 5 mm is the $K_1$ diurnal tide (Doodson number 165.555). It produces a correction of about 13 mm. A few other diurnal tides produce second-order radial corrections which vary from a fraction of a millimeter to 1.8 mm. Their sum is about 4 mm, which is just under the criterion and is ignored. A few long-period tides produce a total radial correction of about 0.4 mm, which is also ignored.

The remainder of this section derives the second-order radial tidal displacement due to the $K_1$ diurnal tide. From Eq. (5–32), the second-order correction to the radial tidal displacement is given by:

$$\Delta s_r = \Delta h_2 \left( \frac{W}{g} \right) \qquad \text{km} \qquad (5\text{–}67)$$

where $W/g$ is the term of Eq. (5–54) for the $K_1$ diurnal ($n = 2$, $m = 1$) tide:

$$\frac{W}{g} = H_{K_1} W_2^1(\phi) \sin\left(\theta_{K_1} + \lambda\right) \qquad \text{km} \qquad (5\text{–}68)$$

From Eqs. (5–55) to (5–59) or from Table 2 on p. 52 of Cartwright and Tayler (1971),

$$W_2^1(\phi) = -\frac{3}{2}\sqrt{\frac{5}{24\,\pi}}\ \sin 2\phi \qquad (5\text{–}69)$$

For the $K_1$ diurnal tide (Doodson argument number 165.555), $n_1 = m = 1$, $n_2 = 1$, and $n_3 = n_4 = n_5 = n_6 = 0$. Hence, from Eqs. (5–60) and (5–63),

$$\theta_{K_1} = \beta_1 + \beta_2 = \theta_{\mathrm{M}} + \pi - s + s = \theta_{\mathrm{M}} + \pi \qquad (5\text{–}70)$$

and

$$\sin\left(\theta_{K_1} + \lambda\right) = \sin\left(\theta_{\mathrm{M}} + \pi + \lambda\right) = -\sin\left(\theta_{\mathrm{M}} + \lambda\right) \qquad (5\text{–}71)$$

From Table 5 on p. 699 of Wahr (1981), the value of $h_2$ for the $K_1$ tide is 0.520. However, p. 57 of International Earth Rotation Service (1992) quotes a value of 0.5203 from Wahr's theory. Using this value and the value of $h_2$ from Eq. (5–35) which is used in computing the first-order tidal displacement from Eq. (5–51) gives:

$$\Delta h_2 = 0.5203 - 0.6090 = -\,0.0887 \tag{5–72}$$

From p. 259 of Cartwright and Edden (1973), the value of the amplitude $H_s$ for the $K_1$ tide is:

$$H_{K_1} = 0.36878 \text{ m} \tag{5–73}$$

From Eqs. (5–67) to (5–73), the second-order term of the radial displacement of the Earth-fixed tracking station due to solid Earth tides is:

$$\begin{aligned}\Delta s_r &= (-\,0.0887)(0.36878 \text{ m})\left(-\frac{3}{2}\right)\sqrt{\frac{5}{24\,\pi}}\sin 2\phi\left[-\sin(\theta_{\mathrm{M}} + \lambda)\right] \\ &= -\left(1.264 \times 10^{-5} \text{ km}\right)\sin 2\phi \sin(\theta_{\mathrm{M}} + \lambda)\end{aligned} \tag{5–74}$$

where $\phi$ and $\lambda$ are the geocentric latitude and longitude of the tracking station, referred to the true pole, prime meridian, and equator of date. However, since this term is so small, $\phi$ and $\lambda$ can be evaluated with the input 1903.0 station coordinates, which are uncorrected for polar motion. Also, as discussed in the previous section, mean sidereal time $\theta_{\mathrm{M}}$ can be replaced with true sidereal time $\theta$, with a resulting error of less than 0.002 mm. For a tracking station with a latitude of $\pm 45°$, the amplitude of $\Delta s_r$ is 1.3 cm. The second form of Eq. (5–74) is given on p. 58 of International Earth Rotation Service (1992).

In Eq. (5–51) for the first-order displacement of the tracking station due to solid Earth tides, the radial, north, and east displacements were computed from phase-shifted values of the unit vector $\hat{\mathbf{r}}$ to the tracking station and the corresponding north **N** and east **E** vectors. This is equivalent to calculating these components of the displacement with the longitude $\lambda$ of the tracking station reduced by the phase shift $\psi$ (see Eqs. 5–47 and 5–48). Although this phase shift was not considered in the expansion of the tidal potential, it can be added by replacing $\lambda$ with $(\lambda - \psi)$ in Eq. (5–54). It follows that this same substitution should be made in Eqs. (5–68), (5–71), and (5–74).

The second-order term of the displacement of the tracking station due to solid Earth tides is obtained by substituting $\Delta s_r$ given by Eq. (5–74) (with $\lambda$ replaced by $\lambda - \psi$) and $\Delta s_\phi = \Delta s_\lambda = 0$ into Eq. (5–36):

$$\Delta \mathbf{r}_{\mathrm{b}} = \Delta s_r \, \hat{\mathbf{r}} \qquad \text{km} \qquad (5\text{–}75)$$

where $\hat{\mathbf{r}}$ is obtained by substituting the first five terms of Eq. (5–1) into Eq. (5–25).

#### 5.2.6.6 Permanent Displacement of the Tracking Station Due to Solid Earth Tides

This section develops the equations for the constant part of the displacement of the tracking station due to solid Earth tides. This permanent tidal displacement is included in the calculated first-order tidal displacement. If the permanent tidal displacement was subtracted from the sum of the first-order and second-order tidal displacements, then the estimated coordinates of the tracking station would include the permanent tidal displacement. However, this calculation is not performed in any of the major orbit determination programs that calculate solid Earth tides. Hence, to be consistent, we will not subtract the permanent tidal displacement from the sum of the first-order and second-order tidal displacements.

The remainder of this section derives the equations for calculating the permanent displacement of the tracking station due to solid Earth tides. However, these equations will not be evaluated. This are given for information only.

The permanent tidal displacement of the tracking station is calculated from Eqs. (5–32) to (5–34), where $W_2/g$ is the zero-frequency term of Eq. (5–54). From Cartwright and Edden (1973), the zero-frequency tide has the Doodson argument number 055.555. This means that $n_1 = m = 0$ and $n_2$ through $n_6$ are zero. Hence, from Eq. (5–60), the astronomical argument $\theta_s$ is zero. Since $n = 2$ and $m = 0$ for the zero-frequency tide,

$$\begin{matrix}\cos\\ \sin\end{matrix}\left(\theta_s + m\lambda\right) = \cos(0) = 1 \tag{5–76}$$

and the zero-frequency term of Eq. (5–54) is:

$$\frac{W}{g} = H_s W_2^0(\phi) \qquad \text{m} \tag{5–77}$$

From Cartwright and Edden (1973), the amplitude $H_s$ for the zero-frequency tide is:

$$H_s = -\,0.31455 \text{ m} \tag{5–78}$$

From Eqs. (5–55) to (5–59), or from Cartwright and Tayler (1971), p. 52,

$$W_2^0(\phi) = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\sin^2\phi - \frac{1}{2}\right) \tag{5–79}$$

and

$$\frac{\partial W_2^0(\phi)}{\partial \phi} = \frac{3}{2}\sqrt{\frac{5}{4\pi}}\,\sin 2\phi \tag{5–80}$$

From Wahr (1981), p. 699, Table 5, the values of the Love numbers $h_2$ and $l_2$ that apply for any long-period tide ($n = 2$, $m = 0$) are:

$$\begin{aligned} h_2 &= 0.606 \\ l_2 &= 0.0840 \end{aligned} \tag{5–81}$$

The actual permanent tide should be computed from these values of $h_2$ and $l_2$. However, if the permanent tide is calculated for the purpose of subtracting it from the first-order tidal displacement calculated from Eq. (5–51) (in order to eliminate the permanent tide that is included in the first-order tidal displacement), then the permanent tide should be computed from $h_2$ and $l_2$ given by Eq. (5–35), since these values were used in Eq. (5–51).

The radial component of the permanent tide at the tracking station is obtained by substituting Eqs. (5–77) to (5–79) into Eq. (5–32):

$$\begin{aligned} s_r &= h_2\left(-\,0.31455\text{ m}\right)\sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\sin^2\phi-\frac{1}{2}\right) \\ &= -\,h_2\left(0.19841\times10^{-3}\text{ km}\right)\left(\frac{3}{2}\sin^2\phi-\frac{1}{2}\right) \end{aligned} \tag{5–82}$$

Substituting the partial derivative of Eq. (5–77) with respect to $\phi$, Eq. (5–78), and Eq. (5–80) into Eq. (5–33) gives the north component of the permanent tide at the tracking station:

$$\begin{aligned} s_\phi &= l_2\left(-\,0.31455\text{ m}\right)\left(\frac{3}{2}\sqrt{\frac{5}{4\pi}}\right)\sin 2\phi \\ &= -\,l_2\left(0.29762\times10^{-3}\text{ km}\right)\sin 2\phi \end{aligned} \tag{5–83}$$

Using the values of $h_2$ and $l_2$ from Eq. (5–35), the coefficients in Eqs. (5–82) and (5–83), which multiply the functions of $\phi$ are –0.12083 m and –0.02536 m, respectively. Eqs. (5–82) and (5–83) with these numerical coefficients, are Eqs. (8a) and (8b) on p. 58 of International Earth Rotation Service (1992). Since Eqs. (5–77) and (5–79) are not a function of the longitude $\lambda$ of the tracking station, the east component of the permanent tide at the tracking station, computed from Eq. (5–34), is zero.

From Eq. (5–36), with the east component $s_\lambda$ set to zero, the permanent displacement of the tracking station due to solid Earth tides is given by:

$$\Delta\mathbf{r}_\mathrm{b} = s_r\,\hat{\mathbf{r}} + s_\phi\,\mathbf{N} \qquad \text{km} \tag{5–84}$$

where $s_r$ and $s_\phi$ are given by Eqs. (5–82) and (5–83). The unit vector $\hat{\mathbf{r}}$ to the tracking station is obtained by substituting the first five terms of Eq. (5–1) into Eq. (5–25). The north vector $\mathbf{N}$ is calculated from Eq. (5–37). The geocentric latitude $\phi$ and longitude $\lambda$ of the tracking station used to evaluate $s_r$, $s_\phi$, and $\mathbf{N}$ can be the input 1903.0 values, which are uncorrected for polar motion. The error due to ignoring polar motion in these calculations is less than 0.001 mm.

### 5.2.7 OCEAN LOADING

The seventh term of Eq. (5–1) is the displacement $\Delta\mathbf{r}_{\mathrm{OL}}$ of a fixed tracking station on Earth due to ocean loading. This is a centimeter-level periodic displacement due to the periodic ocean tides. It is calculated from the model of Scherneck (1991). The displacements in the geocentric radial, north, and east directions (on a spherical Earth) are given by:

$$s_r = +10^{-3} \sum_{s=1}^{11} A_s^r \cos\left(\theta_s + \chi_s - \phi_s^r\right) \qquad \text{km} \qquad (5\text{–}85)$$

$$s_\phi = -10^{-3} \sum_{s=1}^{11} A_s^{\mathrm{S}} \cos\left(\theta_s + \chi_s - \phi_s^{\mathrm{S}}\right) \qquad \text{km} \qquad (5\text{–}86)$$

$$s_\lambda = -10^{-3} \sum_{s=1}^{11} A_s^{\mathrm{W}} \cos\left(\theta_s + \chi_s - \phi_s^{\mathrm{W}}\right) \qquad \text{km} \qquad (5\text{–}87)$$

where $A_s^r$, $A_s^{\mathrm{S}}$, and $A_s^{\mathrm{W}}$ are the amplitudes (in meters) of the radial, south, and west displacements for tide $s$. The astronomical argument $\theta_s$ for tide $s$ is calculated from the Doodson argument number, Eq. (5–60), and related equations as described in Sections 5.2.6.3 and 5.2.6.4. The quantity $\chi_s$ is the additional Schwiderski phase angle, which will be discussed below. The angles $\phi_s^r$, $\phi_s^{\mathrm{S}}$, $\phi_s^{\mathrm{W}}$ (which are given in degrees) are the Greenwich phase lags for the radial, south, and west displacements for tide $s$. The summations are over eleven tide components: the $M_2$, $S_2$, $N_2$, and $K_2$ semi-diurnal tides; the $K_1$, $O_1$, $P_1$, and $Q_1$ diurnal tides; and the $M_{\mathrm{f}}$, $M_{\mathrm{m}}$, and $S_{\mathrm{sa}}$ long-period tides. The Doodson argument number and the corresponding values of the integers $n_1$ through $n_6$ for each of these tides are shown in Table 5-1.

**Table 5–1**

**Doodson Argument Numbers**

| Tide | Doodson Argument Number | $n_1$ | $n_2$ | $n_3$ | $n_4$ | $n_5$ | $n_6$ |
|---|---|---|---|---|---|---|---|
| $M_2$ | 255.555 | 2 | 0 | 0 | 0 | 0 | 0 |
| $S_2$ | 273.555 | 2 | 2 | –2 | 0 | 0 | 0 |
| $N_2$ | 245.655 | 2 | –1 | 0 | 1 | 0 | 0 |
| $K_2$ | 275.555 | 2 | 2 | 0 | 0 | 0 | 0 |
| $K_1$ | 165.555 | 1 | 1 | 0 | 0 | 0 | 0 |
| $O_1$ | 145.555 | 1 | –1 | 0 | 0 | 0 | 0 |
| $P_1$ | 163.555 | 1 | 1 | –2 | 0 | 0 | 0 |
| $Q_1$ | 135.655 | 1 | –2 | 0 | 1 | 0 | 0 |
| $M_\mathrm{f}$ | 075.555 | 0 | 2 | 0 | 0 | 0 | 0 |
| $M_\mathrm{m}$ | 065.455 | 0 | 1 | 0 | –1 | 0 | 0 |
| $S_\mathrm{sa}$ | 057.555 | 0 | 0 | 2 | 0 | 0 | 0 |

From International Earth Rotation Service (1992), p. 63, Table 8.1, the additional Schwiderski phase angle $\chi_s$ is a function of the tide period band (*i.e.*, semi-diurnal, diurnal, or long-period) and the sign of the amplitude $H_s$ of the tide (see Eq. 5–54):

$$\chi_s = \begin{bmatrix} 0 \\ 0 \\ \pi/2 \\ -\pi/2 \end{bmatrix} \begin{array}{l} \text{Semi-Diurnal Tides with } H_s > 0\ (M_2,\ S_2,\ N_2,\ K_2) \\ \text{Long-Period Tides with } H_s < 0\ (M_\mathrm{f},\ M_\mathrm{m},\ S_\mathrm{sa}) \\ \text{Diurnal Tides with } H_s > 0\ (K_1) \\ \text{Diurnal Tides with } H_s < 0\ (O_1,\ P_1,\ Q_1) \end{array} \tag{5–88}$$

Calculation of the displacement of a tracking station due to ocean loading requires the three amplitudes $A_s^r$, $A_s^{\mathrm{S}}$, and $A_s^{\mathrm{W}}$ and the three phases $\phi_s^r$, $\phi_s^{\mathrm{S}}$, $\phi_s^{\mathrm{W}}$ for each of the eleven tide components (a total of 66 numbers) which apply for that tracking station location. Pages 70–109 of International Earth Rotation Service (1992), contain tables of these 66 ocean-loading coefficients which apply for a large number of locations on Earth. We use the table labelled MOJAVE12 for each tracking station at the Goldstone complex, the table labelled TIDBIN64 for each tracking station at the Canberra, Australia complex, and the table labelled MADRID64 for each tracking station at the Madrid, Spain complex.

The Earth-fixed displacement vector $\Delta\mathbf{r}_{\mathrm{OL}}$ of a fixed tracking station on Earth due to ocean loading is calculated by substituting the geocentric radial, north, and east displacements calculated from Eqs. (5–85) to (5–87) into Eq. (5–36). The unit vector $\hat{\mathbf{r}}$ to the tracking station is calculated by substituting the first five terms of Eq. (5–1) into Eq. (5–25). The north **N** and east **E** vectors can be calculated from Eqs. (5–37) and (5–38) using input 1903.0 station coordinates, which are uncorrected for polar motion.

### 5.2.8 POLE TIDE

The eighth term of Eq. (5–1) is the displacement $\Delta\mathbf{r}_{\mathrm{PT}}$ of a fixed tracking station on Earth due to the so-called pole tide. This is a solid Earth tide caused by polar motion. The equations for calculating the pole tide are derived in Section 5.2.8.1. It will be seen that the components of the pole tide are proportional to $X - \overline{X}$ and $Y - \overline{Y}$, where $X$ and $Y$ are the Earth-fixed coordinates of the true pole of date relative to the mean pole of 1903.0. The quantities $\overline{X}$ and $\overline{Y}$ are average values of $X$ and $Y$ over some modern time span. Section 5.2.8.2 derives equations for constant values of the Earth's normalized harmonic coefficients $\overline{C}_{21}$ and $\overline{S}_{21}$ as functions of $\overline{X}$ and $\overline{Y}$. These equations are inverted to give the required values of $\overline{X}$ and $\overline{Y}$ as functions of $\overline{C}_{21}$ and $\overline{S}_{21}$. These are not the estimated values of the Earth's harmonic coefficients. They are constant values obtained from the GIN file, which are only used in the pole tide model in program Regres.

The deformation of the Earth due to the pole tide produces periodic changes in the Earth's normalized harmonic coefficients $\overline{C}_{21}$ and $\overline{S}_{21}$. The equations for calculating these periodic terms are derived in Section 5.2.8.3. The periodic variations in $\overline{C}_{21}$ and $\overline{S}_{21}$ are added to the estimated values of $\overline{C}_{21}$ and $\overline{S}_{21}$ in program PV. Calculation of the periodic variations requires values of $\overline{X}$ and $\overline{Y}$, which are calculated from the equations of Section 5.2.8.2 as functions of the estimated harmonic coefficients $\overline{C}_{21}$ and $\overline{S}_{21}$ instead of the constant values used in program Regres.

### 5.2.8.1 Derivation of Equations for the Pole Tide

This section derives the equations for calculating the displacement of the tracking station due to the deformation of the Earth caused by polar motion. The displacement of a tracking station due to this effect is less than 2 cm. The derivation given here was taken from Wahr (1985).

From p. 4, Eq. (5) of Melbourne *et al*. (1968), the geoid (mean sea level) is an equipotential surface, where the potential is the sum of the gravitational potential and the centrifugal potential. Polar motion changes the centrifugal potential and thus the geoid. The Earth-fixed rectangular coordinate system used to derive the pole tide is aligned with the mean pole, prime meridian, and equator of 1903.0. From Eq. (1) of Wahr (1985), the instantaneous angular rotation vector of the Earth, with rectangular components in the Earth-fixed 1903.0 coordinate system, is given by:

$$\Omega = \omega_{\mathrm{E}} \begin{bmatrix} X \\ -Y \\ 1 \end{bmatrix} \qquad \text{rad/s} \qquad (5\text{–}89)$$

where terms quadratic in $X$ and $Y$ and variations in the Earth's rotation rate are ignored. The mean inertial rotation rate of the Earth ($\omega_{\mathrm{E}}$) is given in Section 4.3.1.2. The quantities $X$ and $Y$ are the angular coordinates (in radians) of the Earth's true pole of date (instantaneous axis of rotation) relative to the mean pole of 1903.0. The angle $X$ is measured south along the $0°$ meridian of 1903.0, and $Y$

is measured south along the 90° W meridian of 1903.0. These angles are interpolated from the EOP file or the TP array as described in Section 5.2.5.1.

From Eq. (2) of Wahr (1985), the centrifugal potential $U_c$ at the location of the tracking station is given by:

$$U_c = \frac{1}{2}\left[r^2|\Omega|^2 - (\mathbf{r}\cdot\Omega)^2\right] \qquad \mathrm{km^2/s^2} \qquad (5\text{–}90)$$

where **r** is the geocentric position vector of the tracking station with rectangular components along the Earth-fixed 1903.0 coordinate system:

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\phi\cos\lambda \\ r\cos\phi\sin\lambda \\ r\sin\phi \end{bmatrix} \qquad \mathrm{km} \qquad (5\text{–}91)$$

where $r$, $\phi$, and $\lambda$ are the geocentric radius, latitude, and longitude of the tracking station in the Earth-fixed 1903.0 coordinate system. Substituting Eq. (5–89) and the first form of Eq. (5–91) into Eq. (5–90) gives a number of terms of $U_c$. The first-order term is the nominal centrifugal potential, which produces the ellipticity of the Earth. All terms quadratic in $X$ and $Y$ are ignored. The sum $V$ of the terms linear in $X$ and $Y$ is the perturbation to the centrifugal potential due to polar motion:

$$V = -\,\omega_E{}^2 z\,(Xx - Yy) \qquad \mathrm{km^2/s^2} \qquad (5\text{–}92)$$

Substituting $x$, $y$, and $z$ from Eq. (5–91) as functions of $r$, $\phi$, and $\lambda$ gives:

$$V = -\frac{1}{2}\omega_E{}^2 r^2 \sin 2\phi\,(X\cos\lambda - Y\sin\lambda) \qquad \mathrm{km^2/s^2} \qquad (5\text{–}93)$$

which is equivalent to Eq. (3) of Wahr (1985). The $X$ and $Y$ coordinates of the true pole of date can be expressed as sums of the mean coordinates $\overline{X}$ and $\overline{Y}$ (which are constant in program Regres) and the periodic variations of the coordinates $X-\overline{X}$ and $Y-\overline{Y}$. The change $V$ in the centrifugal potential due to the displacement of the mean pole $(\overline{X}, \overline{Y})$ from the 1903.0 pole produces constant changes in the coordinates of the tracking stations, which can be absorbed into the input 1903.0 coordinates. The displacement of the tracking station due to the displacement of the true pole of date $(X,Y)$ from the mean pole $(\overline{X}, \overline{Y})$ is calculated from the potential:

$$V = -\frac{1}{2}\omega_{\mathrm{E}}{}^2 r^2 \sin 2\phi \left[\left(X - \overline{X}\right)\cos\lambda - \left(Y - \overline{Y}\right)\sin\lambda\right] \qquad \mathrm{km}^2/\mathrm{s}^2 \qquad (5\text{–}94)$$

The displacements of the tracking station in the radial, north, and east directions due to the change $V$ in the centrifugal potential due to the periodic terms of polar motion are obtained by substituting $V$ given by Eq. (5–94) for $W_2$ in Eqs. (5–32) to (5–34):

$$s_r = -\frac{h_2}{2}\frac{\omega_{\mathrm{E}}{}^2 r^2}{g}\sin 2\phi \left[\left(X - \overline{X}\right)\cos\lambda - \left(Y - \overline{Y}\right)\sin\lambda\right] \qquad \mathrm{km} \qquad (5\text{–}95)$$

$$s_\phi = -\, l_2 \frac{\omega_{\mathrm{E}}{}^2 r^2}{g}\cos 2\phi \left[\left(X - \overline{X}\right)\cos\lambda - \left(Y - \overline{Y}\right)\sin\lambda\right] \qquad \mathrm{km} \qquad (5\text{–}96)$$

$$s_\lambda = +\, l_2 \frac{\omega_{\mathrm{E}}{}^2 r^2}{g}\sin\phi \left[\left(X - \overline{X}\right)\sin\lambda + \left(Y - \overline{Y}\right)\cos\lambda\right] \qquad \mathrm{km} \qquad (5\text{–}97)$$

where $g$ is the acceleration of gravity at the tracking station. An approximate value which can be used at all tracking stations will be given below. The Love numbers $h_2$ and $l_2$ should be the long-period values given in Eq. (5–81). However, the only available values are the input semi-diurnal values given by Eq. (5–35). Use of these values in Eqs. (5–95) to (5–97) produces errors of 0.1 mm or less. The displacement $\Delta\mathbf{r}_{\mathrm{PT}}$ of the tracking station due to the pole tide is obtained by substituting $s_r$, $s_\phi$, and $s_\lambda$ calculated from Eqs. (5–95) to (5–97) into Eq. (5–36). In this equation, $\hat{\mathbf{r}}$ is obtained by substituting the first five terms of Eq. (5–1) into Eq. (5–25). The north **N** and east **E** vectors are calculated from Eqs. (5–37) and (5–38). The spherical coordinates $r$, $\phi$, and $\lambda$ of the tracking station used in Eqs. (5–95) to (5–97), Eq. (5–37), and Eq. (5–38) can be the input 1903.0

coordinates, uncorrected for polar motion. The pole tide displacement should be referred to the true pole, prime meridian, and equator of date. However, most of the calculated quantities are referred to the mean pole, prime meridian, and equator of 1903.0. The resulting errors are negligible because the displacement is less than 2 cm.

Page 700 of Explanatory Supplement (1992) gives an expression for the acceleration of gravity $g$ as a function of the latitude $\phi$. This expression is an even function of $\phi$. The three DSN complexes have absolute latitudes of 35°, 35°, and 40°. There are a number of other stations which have smaller absolute latitudes. The acceleration of gravity $g$ is approximately 9.78 m/s$^2$ at $\phi = 0°$, 9.80 m/s$^2$ at $\phi = 38°$, 9.82 m/s$^2$ at $\phi = 61°$, and 9.832 m/s$^2$ at $\phi = 90°$. For the pole tide model, we will set $g$ equal to the constant value of 9.80 m/s$^2$:

$$g = 9.80 \times 10^{-3} \text{ km / s}^2 \tag{5–98}$$

For a tracking station at any latitude, the maximum error in $g$ given by Eq. (5–98) is 0.33%. The corresponding error in a 2 cm pole tide would be less than 0.1 mm.

#### 5.2.8.2 Calculation of the Mean Position $\left(\overline{X}, \overline{Y}\right)$ of the True Pole $(X, Y)$

This section develops equations that can be used to calculate the mean values $\overline{X}$ and $\overline{Y}$ of the $X$ and $Y$ coordinates of the true pole of date. They are used in Eqs. (5–95) to (5–97) to calculate the radial, north, and east displacements of the tracking station due to the pole tide. They are also required in the equations of the following section for the periodic variations in the Earth's normalized harmonic coefficients $\overline{C}_{21}$ and $\overline{S}_{21}$. These periodic terms are due to the deformation of the Earth caused by the pole tide.

In the Earth-fixed coordinate system aligned with the mean pole, prime meridian, and equator of 1903.0, the current mean pole is not aligned with the z axis but is located $\overline{X}$ radians south along the Greenwich meridian and $\overline{Y}$ radians south along the 90° W meridian. From p. 42 of International Earth Rotation Service (1992), it is assumed that the Earth's mean figure axis has the same orientation as the mean rotation pole, when averaged over the same long time period. Hence, the Earth-fixed coordinates of the mean figure axis are $\left(\overline{X}, \overline{Y}\right)$. At a fixed point in the 1903.0 Earth-fixed coordinate system with geocentric radius $r$, latitude $\phi$, and east longitude $\lambda$, the displacement $\left(\overline{X}, \overline{Y}\right)$ of the current mean pole and figure axis from the 1903.0 mean pole changes the latitude by (see Moyer (1971), Eq. 220):

$$\Delta\phi = \overline{X}\cos\lambda - \overline{Y}\sin\lambda \qquad \text{rad} \qquad (5\text{–}99)$$

In calculating the change in the Earth's gravitational potential due to the change $\Delta\phi$ in the latitude, the gravitational potential $U$ can be approximated with the potential due to the second zonal harmonic $J_2$. From Moyer (1971), Eqs. (158) and (175) to (177), it is given by:

$$U(J_2) = -\frac{\mu_{\mathrm{E}}}{r} J_2 \left(\frac{a_{\mathrm{e}}}{r}\right)^2 \left(\frac{3}{2}\sin^2\phi - \frac{1}{2}\right) \qquad \mathrm{km}^2/\mathrm{s}^2 \qquad (5\text{–}100)$$

The change in this potential due to moving the mean figure axis from the z axis to the point $\left(\overline{X}, \overline{Y}\right)$ is obtained by differentiating Eq. (5–100) with respect to $\phi$ and then multiplying the result by $\Delta\phi$ given by Eq. (5–99):

$$\Delta U = -\frac{\mu_{\mathrm{E}}}{r} J_2 \left(\frac{a_{\mathrm{e}}}{r}\right)^2 \left(\frac{3}{2}\sin 2\phi\right)\left(\overline{X}\cos\lambda - \overline{Y}\sin\lambda\right) \qquad \mathrm{km}^2/\mathrm{s}^2 \qquad (5\text{–}101)$$

This potential has the same form as the potential due to the harmonic coefficients $C_{21}$ and $S_{21}$ (see Moyer (1971), Eqs. 159 and 155):

$$U = \frac{\mu_{\mathrm{E}}}{r} \left(\frac{a_{\mathrm{e}}}{r}\right)^2 \left(\frac{3}{2}\sin 2\phi\right)\left(C_{21}\cos\lambda + S_{21}\sin\lambda\right) \qquad \mathrm{km}^2/\mathrm{s}^2 \qquad (5\text{–}102)$$

Equating (5–101) and (5–102) gives the following approximate additions to the Earth's harmonic coefficients due to the offset $\left(\overline{X}, \overline{Y}\right)$ of the current mean pole and figure axis from the 1903.0 mean pole:

$$\begin{aligned} C_{21} &= -J_2\,\overline{X} \\ S_{21} &= +J_2\,\overline{Y} \end{aligned} \tag{5–103}$$

From p. 54 of International Earth Rotation Service (1992), the unnormalized harmonic coefficients in (5–103) are related to the corresponding normalized coefficients by:

$$\begin{aligned} C_{21} &= N_{21}\,\overline{C}_{21} \\ S_{21} &= N_{21}\,\overline{S}_{21} \\ J_2 = -C_{20} &= -N_{20}\,\overline{C}_{20} = N_{20}\,\overline{J}_2 \end{aligned} \tag{5–104}$$

where

$$N_{nm} = \left[\frac{(n-m)!\,(2n+1)\,(2-\delta_{0m})}{(n+m)!}\right]^{\frac{1}{2}} \tag{5–105}$$

Evaluating $N_{21}$ and $N_{20}$ gives:

$$\begin{aligned} N_{21} &= \sqrt{\frac{5}{3}} \\ N_{20} &= \sqrt{5} \end{aligned} \tag{5–106}$$

Substituting (5–104) and (5–106) into (5–103) gives:

$$\begin{aligned} \overline{C}_{21} &= -\sqrt{3}\,\overline{J}_2\,\overline{X} \\ \overline{S}_{21} &= +\sqrt{3}\,\overline{J}_2\,\overline{Y} \end{aligned} \tag{5–107}$$

Inverting these equations gives the required expressions for calculating the mean values $\left(\overline{X}, \overline{Y}\right)$ of the $X$ and $Y$ coordinates of the true pole of date:

$$\overline{X} = -\frac{\overline{C}_{21}}{\sqrt{3}\,\bar{J}_2}$$
$$\overline{Y} = +\frac{\overline{S}_{21}}{\sqrt{3}\,\bar{J}_2} \qquad (5\text{–}108)$$

From p. 43 of International Earth Rotation Service (1992), the recommended values of $\overline{C}_{21}$ and $\overline{S}_{21}$ are:

$$\overline{C}_{21} = -0.17\times10^{-9}$$
$$\overline{S}_{21} = +1.19\times10^{-9} \qquad (5\text{–}109)$$

These values are GIN file inputs, which are used in program Regres only to calculate $\overline{X}$ and $\overline{Y}$ from Eqs. (5–108). Given the value of $J_2$ from Section 4.3.1.2, the required value of $\bar{J}_2$ can be calculated from Eqs. (5–104) and (5–106). The result is $\bar{J}_2 = 4.8417\times10^{-4}$.

#### 5.2.8.3 Periodic Variations in $\overline{C}_{21}$ and $\overline{S}_{21}$

The change $V$ in the centrifugal potential at the location of a tracking station on Earth due to the periodic part of the polar motion is given by Eq. (5–94). The displacement of the Earth at this point due to $V$ is given by Eqs. (5–95) to (5–97). The induced gravitational potential at the tracking station due to this displacement is the potential $V$ multiplied by the second-degree Love number $k_2$. The induced potential $k_2V$ has very nearly the same form on the Earth's surface as the gravitational potential $U$ due to the Earth's harmonic coefficients $C_{21}$ and $S_{21}$ (Eq. 5–102). Equating $k_2V$ to $U$ at the Earth's surface and converting from unnormalized to normalized harmonic coefficients using Eqs. (5–104) and (5–106) gives the following equations for the periodic variations in $\overline{C}_{21}$ and $\overline{S}_{21}$:

$$\delta\overline{C}_{21} = -K\left(X-\overline{X}\right)$$
$$\delta\overline{S}_{21} = +K\left(Y-\overline{Y}\right) \qquad (5\text{–}110)$$

where

$$K = \frac{\omega_E^{\,2}\, r^5\, k_2}{\mu_E\, a_e^{\,2} \sqrt{15}} \approx \frac{\omega_E^{\,2}\, a_e^{\,3}\, k_2}{\mu_E \sqrt{15}} \tag{5–111}$$

For an accuracy of 9 x $10^{-12}$ in the Earth's normalized harmonic coefficients, the variation in $K$ given by the first form of Eq. (5–111) due to the variation of the geocentric radius $r$ with latitude can be ignored and $K$ can be computed from the second form of (5–111). Substituting numerical values obtained from Section 4.3.1.2 gives:

$$K = \left(8.9373 \times 10^{-4}\right) k_2 \tag{5–112}$$

which should be evaluated with the input value of the second-degree Love number $k_2$. Using the nominal value of 0.30 for $k_2$, $K$ = 2.68 x $10^{-4}$.

Eqs. (5–110) and (5–112) should be used in program PV to calculate periodic corrections to the input or estimated values of the Earth's normalized harmonic coefficients $\overline{C}_{21}$ and $\overline{S}_{21}$. The required values for $\overline{X}$ and $\overline{Y}$ can be computed from the input or estimated values of $\overline{C}_{21}$, $\overline{S}_{21}$, and $\overline{J}_2$ using Eqs. (5–108). In program PV, these harmonic coefficients can be linear functions of time.

## 5.3 EARTH-FIXED TO SPACE-FIXED TRANSFORMATION MATRIX $T_E$ AND ITS TIME DERIVATIVES

This section gives the formulation for the Earth-fixed to space-fixed transformation matrix $T_E$ and its first and second time derivatives with respect to coordinate time ET. Subsection 5.3.1 gives the high-level equations for $T_E$, its time derivatives, and partial derivatives with respect to solve-for parameters. Calculation of the rotation matrix $T_E$ requires the nutation angles and their time derivatives, Universal Time UT1, and (in program PV) the $X$ and $Y$ coordinates of the pole. The procedures for obtaining these quantities are described in Subsection 5.3.2. If the input values of UT1 are regularized (*i.e.,* UT1R), then periodic variations (ΔUT1) in UT1 must be added to UT1R to convert it to UT1. The formulation for calculating ΔUT1 is given in Subsection 5.3.3. Subsections

5.3.4 through 5.3.6 give the formulations for calculating the various sub-matrices of $T_E$, their time derivatives, and partial derivatives with respect to solve-for parameters. The final expressions for the partial derivatives of $T_E$ and the geocentric space-fixed position vector of the tracking station with respect to solve-for parameters will be given in Section 5.5.

### 5.3.1 HIGH-LEVEL EQUATIONS FOR $T_E$, ITS TIME DERIVATIVES, AND PARTIAL DERIVATIVES

The Earth-fixed to space-fixed transformation matrix $T_E$ is used to transform the geocentric Earth-fixed position vector $\mathbf{r}_b$ of a tracking station to the corresponding space-fixed position vector $\mathbf{r}_{TS}^{E}$ of the tracking station (TS) relative to the Earth (E):

$$\mathbf{r}_{TS}^{E} = T_E \, \mathbf{r}_b \qquad \text{km} \qquad (5\text{–}113)$$

The geocentric Earth-fixed position vector $\mathbf{r}_b$ of the tracking station has rectangular components referred to the true pole, prime meridian, and equator of date. The geocentric space-fixed position vector $\mathbf{r}_{TS}^{E}$ of the tracking station has rectangular components that are represented in the celestial reference frame of the particular planetary ephemeris used by the ODP (see Section 3.1.1). Each of the various celestial reference frames is a rectangular coordinate system nominally aligned with the mean Earth equator and equinox of J2000 (see Section 2.1). The celestial reference frame of the planetary ephemeris can have a slightly different orientation for each planetary ephemeris. The celestial reference frame maintained by the International Earth Rotation Service (IERS) is called the radio frame. The right ascensions and declinations of quasars are referred to the radio frame. The transformation matrix $T_E$ rotates from the Earth-fixed coordinate system to the space-fixed radio frame and then to the space-fixed planetary ephemeris frame (which for some ephemerides is the radio frame).

From Eq. (5–113), the transformation from space-fixed to Earth-fixed coordinates of a tracking station is given by:

$$\mathbf{r}_{\mathrm{b}} = T_{\mathrm{E}}{}^{\mathrm{T}}\, \mathbf{r}_{\mathrm{TS}}^{\mathrm{E}} \qquad \mathrm{km} \tag{5–114}$$

where the superscript T indicates the transpose of the matrix.

The Earth-fixed to space-fixed transformation matrix $T_{\mathrm{E}}$ used in program Regres of the ODP is the transpose of the product of six coordinate system rotation matrices:

$$T_{\mathrm{E}} = \left(B\,N\,A\,R_{\mathrm{x}}R_{\mathrm{y}}R_{\mathrm{z}}\right)^{\mathrm{T}} \tag{5–115}$$

The transpose of this matrix is the space-fixed to Earth-fixed transformation matrix $T_{\mathrm{E}}{}^{\mathrm{T}}$:

$$T_{\mathrm{E}}{}^{\mathrm{T}} = \left(B\,N\,A\,R_{\mathrm{x}}R_{\mathrm{y}}R_{\mathrm{z}}\right) \tag{5–116}$$

The definitions of the rotation matrices in Eqs. (5–115) and (5–116) are easier to comprehend if we consider the rotation (5–116) from space-fixed to Earth-fixed coordinates. Starting from the space-fixed coordinate system of the planetary ephemeris, the rotation matrix $R_{\mathrm{z}}$ is a rotation of this coordinate system about its z axis through the small angle $r_{\mathrm{z}}$:

$$R_{\mathrm{z}} = \begin{bmatrix} \cos r_{\mathrm{z}} & \sin r_{\mathrm{z}} & 0 \\ -\sin r_{\mathrm{z}} & \cos r_{\mathrm{z}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5–117}$$

Then, the resulting coordinate system is rotated about its y axis through the small angle $r_{\mathrm{y}}$:

$$R_{\mathrm{y}} = \begin{bmatrix} \cos r_{\mathrm{y}} & 0 & -\sin r_{\mathrm{y}} \\ 0 & 1 & 0 \\ \sin r_{\mathrm{y}} & 0 & \cos r_{\mathrm{y}} \end{bmatrix} \tag{5–118}$$

The resulting coordinate system is rotated about its x axis through the small angle $r_x$:

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos r_x & \sin r_x \\ 0 & -\sin r_x & \cos r_x \end{bmatrix} \tag{5–119}$$

The rotation $R_x R_y R_z$ rotates space-fixed coordinates from the planetary ephemeris frame to the radio frame. The constant rotation angles $r_z$, $r_y$, and $r_x$ can be different for each planetary ephemeris. In order to estimate values of these angles or to consider the effects of their uncertainties on the estimates of other parameters, we will need partial derivatives of observed quantities with respect to these angles. The derivatives of $R_z$, $R_y$, and $R_x$ with respect to $r_z$, $r_y$, and $r_x$, respectively, are given by:

$$\frac{dR_z}{dr_z} = \begin{bmatrix} -\sin r_z & \cos r_z & 0 \\ -\cos r_z & -\sin r_z & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5–120}$$

$$\frac{dR_y}{dr_y} = \begin{bmatrix} -\sin r_y & 0 & -\cos r_y \\ 0 & 0 & 0 \\ \cos r_y & 0 & -\sin r_y \end{bmatrix} \tag{5–121}$$

$$\frac{dR_x}{dr_x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin r_x & \cos r_x \\ 0 & -\cos r_x & -\sin r_x \end{bmatrix} \tag{5–122}$$

In Eq. (5–116), the precession matrix $A$ rotates from coordinates referred to the mean Earth equator and equinox of J2000 (specifically, the radio frame) to coordinates referred to the mean Earth equator and equinox of date. The nutation matrix $N$ rotates from coordinates referred to the mean Earth equator and equinox of date to coordinates referred to the true Earth equator and equinox of date. The matrix $B$ rotates from space-fixed coordinates referred to the true Earth equator and equinox of date to Earth-fixed coordinates referred to the true pole, prime meridian, and equator of date.

The Earth-fixed to space-fixed transformation matrix $T_{E_{PV}}$ used in program PV rotates from Earth-fixed rectangular coordinates referred to the mean pole, prime meridian, and equator of 1903.0 to space-fixed rectangular coordinates of the planetary ephemeris frame. It is obtained from the rotation matrix $T_E$ used in program Regres by adding an additional rotation matrix:

$$T_{E_{PV}} = \left(P\,B\,N\,A\,R_x R_y R_z\right)^T \tag{5–123}$$

$$T_{E_{PV}}{}^T = \left(P\,B\,N\,A\,R_x R_y R_z\right) \tag{5–124}$$

The polar motion rotation matrix $P$ rotates from Earth-fixed coordinates referred to the true pole, prime meridian, and equator of date to Earth-fixed coordinates referred to the mean pole, prime meridian, and equator of 1903.0. From Eq. (5–15), the polar motion rotation matrix $P$ is defined to be:

$$P^T = R_x(Y)\,R_y(X) \tag{5–125}$$

where $X$ and $Y$ are the angular coordinates of the true pole of date relative to the mean pole of 1903.0, and the two rotation matrices are defined by Eqs. (5–16) and (5–17). Eq. (5–125) is evaluated using the first-order approximations: $\cos X = \cos Y = 1$, $\sin X = X$, and $\sin Y = Y$. In the product of the two rotation matrices, the second-order term $XY$ is ignored. The resulting expression for the polar motion rotation matrix $P$ is given by:

$$P = \begin{bmatrix} 1 & 0 & X \\ 0 & 1 & -Y \\ -X & Y & 1 \end{bmatrix} \tag{5–126}$$

The derivative of $P$ with respect to coordinate time ET is given by:

$$\dot{P} = \begin{bmatrix} 0 & 0 & \dot{X} \\ 0 & 0 & -\dot{Y} \\ -\dot{X} & \dot{Y} & 0 \end{bmatrix} \tag{5–127}$$

where the dots denote time derivatives.

From Eq. (5–115), the derivative of $T_E$ with respect to coordinate time ET is given by:

$$\dot{T}_E = \left[ \left( \dot{B} N A + B \dot{N} A + B N \dot{A} \right) R_x R_y R_z \right]^T \qquad \text{rad/s} \tag{5–128}$$

The second time derivative of $T_E$ can be evaluated from the approximation:

$$\ddot{T}_E = \left( \ddot{B} N A R_x R_y R_z \right)^T \qquad \text{rad/s}^2 \tag{5–129}$$

The formulation for calculating the rotation matrix $B$ and its time derivatives will be given in Subsection 5.3.6. That section will give a simple algorithm for evaluating $\ddot{T}_E$.

The modified nutation-precession matrix $(N A)'$, which is a sub-matrix of Eq. (5–116), is used throughout program Regres:

$$(N A)' = N A R_x R_y R_z \tag{5–130}$$

Its time derivative is given by:

$$\left[ (N A)' \right]^{\cdot} = \left( \dot{N} A + N \dot{A} \right) R_x R_y R_z \qquad \text{rad/s} \tag{5–131}$$

The partial derivatives of $T_E$ with respect to the so-called frame-tie rotation angles $r_z$, $r_y$, and $r_x$ are given by:

$$\frac{\partial T_{\mathrm{E}}}{\partial r_{\mathrm{z}}} = \left( B\,N\,A\,R_{\mathrm{x}} R_{\mathrm{y}} \frac{dR_{\mathrm{z}}}{dr_{\mathrm{z}}} \right)^{\mathrm{T}} \tag{5–132}$$

$$\frac{\partial T_{\mathrm{E}}}{\partial r_{\mathrm{y}}} = \left( B\,N\,A\,R_{\mathrm{x}} \frac{dR_{\mathrm{y}}}{dr_{\mathrm{y}}} R_{\mathrm{z}} \right)^{\mathrm{T}} \tag{5–133}$$

$$\frac{\partial T_{\mathrm{E}}}{\partial r_{\mathrm{x}}} = \left( B\,N\,A \frac{dR_{\mathrm{x}}}{dr_{\mathrm{x}}} R_{\mathrm{y}} R_{\mathrm{z}} \right)^{\mathrm{T}} \tag{5–134}$$

which use Eqs. (5–120) to (5–122).

From Eqs. (5–115) and (5–130), the partial derivative of $T_{\mathrm{E}}$ with respect to Universal Time UT1 is given by:

$$\frac{\partial T_{\mathrm{E}}}{\partial \mathrm{UT1}} = \left[ \frac{\partial B}{\partial \mathrm{UT1}} \left( N\,A \right)' \right]^{\mathrm{T}} \qquad \mathrm{rad/s} \tag{5–135}$$

The partial derivative of the rotation matrix $B$ with respect to UT1 will be given in Subsection 5.3.6. Eq. (5–135) will be used in Section 5.5 to calculate the partial derivative of the space-fixed position vector of the tracking station with respect to UT1.

### 5.3.2 OBTAINING NUTATION ANGLES, UNIVERSAL TIME UT1, AND COORDINATES OF THE POLE

The time argument for calculating the Earth-fixed to space-fixed transformation matrix $T_{\mathrm{E}}$ is coordinate time ET of the Solar-System barycentric or local geocentric space-time frame of reference. In addition to the time argument ET, calculation of the rotation matrix $T_{\mathrm{E}}$ also requires the nutation angles and their time derivatives, Universal Time UT1, and (in program PV) the $X$ and $Y$ coordinates of the pole. This section explains how these additional quantities are obtained.

1. Calculation of several of the auxiliary quantities requires that the time argument ET be transformed to Coordinated Universal Time UTC, which is the argument for the TP array or the EOP file (see section 2.4). This time transformation can be performed using the complete expression for the time difference ET – TAI in the Solar-System barycentric frame or an approximate expression. The expression used will be specified in each application below. In the Solar-System barycentric frame of reference, the complete expression for ET – TAI at a tracking station on Earth is given by Eq. (2–23). However, the geocentric space-fixed position vector of the tracking station $\mathbf{r}_A^E$ can be evaluated with the approximate algorithm given in Section 5.3.6.3. The approximate expression for ET – TAI at a tracking station on Earth in the Solar-System barycentric frame of reference is given by Eqs. (2–26) to (2–28). In the local geocentric space-time frame of reference, ET – TAI at a tracking station on Earth is given by Eq. (2–30). Subtract ET – TAI from the argument ET to give TAI. Use it as the argument to interpolate the TP array or the EOP file for TAI – UTC and subtract it from TAI to give the first value of UTC. Use it as the argument to re-interpolate the TP array or the EOP file for TAI – UTC and subtract it from TAI to give the final value of UTC. At the time of a leap second, the two values of UTC may differ by exactly one second.

2. Using ET as the argument, obtain the nutation in longitude ($\Delta\psi$) and the nutation in obliquity ($\Delta\varepsilon$) in radians and their time derivatives in radians per second:

$$\Delta\psi,\ \Delta\varepsilon,\ (\Delta\psi)^{\cdot},\ \text{and}\ (\Delta\varepsilon)^{\cdot} \tag{5–136}$$

They can be interpolated from the planetary ephemeris, or they can be evaluated directly from the theory of nutation in program GIN. We currently use the 1980 IAU Theory of Nutation, which is given in Seidelmann (1982).

3. Transform the argument ET to UTC using the approximate expression for ET – TAI in the Solar-System barycentric frame. Using UTC as the argument, interpolate the EOP file for the corrections to the nutation angles and their time derivatives:

$$\delta\psi,\ \delta\varepsilon,\ (\delta\psi)^{\cdot},\ \text{and}\ (\delta\varepsilon)^{\cdot} \tag{5–137}$$

   Add the corrections (5–137) to the values (5–136) obtained from the 1980 IAU Theory of Nutation (Seidelmann 1982).

4. In program PV, transform the argument ET to UTC using the approximate expression for ET – TAI in the Solar-System barycentric frame. Using UTC as the argument, interpolate the EOP file or the TP array for the $X$ and $Y$ coordinates of the true pole of date relative to the mean pole of 1903.0 and their time derivatives $\dot{X}$ and $\dot{Y}$.

5. In program Regres, transform the argument ET to UTC using the complete expression for ET – TAI in the Solar-System barycentric frame, as described above in item 1. In program PV, use the approximate expression for ET – TAI in the Solar-System barycentric frame. Using UTC as the argument, interpolate the TP array or the EOP file for TAI – UT1 and its time derivative:

$$\text{TAI} - \text{UT1, and } (\text{TAI} - \text{UT1})^{\cdot} \tag{5–138}$$

   Subtract TAI – UT1 from TAI to give Universal Time UT1. This will be Universal Time UT1 or Regularized Universal Time UT1R. If it is the latter, then the periodic terms (ΔUT1) of UT1 must be calculated from the algorithm given in Section 5.3.3 and added to UT1R to give UT1. In either case, the value of UT1 will be used in Section 5.3.6 to calculate sidereal time $\theta$ and the rotation matrix $B$. The time derivative $(\text{TAI} - \text{UT1})^{\cdot}$ will be used in Section 5.3.6 to calculate $\dot{\theta}$, the time derivative of $\theta$ with respect to coordinate time ET.

## 5.3.3 ALGORITHM FOR PERIODIC TERMS OF UT1

### 5.3.3.1 Introduction

Periodic variations in Universal Time UT1 are derived by Yoder *et al.* (1981). There are 41 short-period terms with periods between 5 and 35 days and 21 long-period terms with periods between 91 and 6791 days. The periodic variations in UT1 are caused by long-period solid Earth tides (having periods greater than those of the various semi-diurnal and diurnal tides) that produce periodic variations in the Earth's polar moment of inertia *C* and hence the angular rotation rate of the Earth.

The time difference TAI – UT1 is obtained by interpolating the TP array or the EOP file. Subtracting TAI – UT1 from TAI gives Universal Time UT1. If it is Regularized Universal Time (UT1R), the sum ΔUT1 of the 41 short-period terms of UT1 was subtracted from the observed values of UT1 before the data was smoothed. For this case, the sum ΔUT1 of the 41 short-period terms of UT1 must be computed from the formulation given in Subsection 5.3.3.2 and added to UT1R to give UT1. If Universal Time obtained from the TP array or the EOP file is not regularized, then no correction is necessary.

Table 5–2 (which will be described in Subsection 5.3.3.2) lists the 41 short-period terms of UT1. The largest amplitude of a single term is about 0.8 ms, which can affect the space-fixed position vector of a tracking station on Earth by about 0.4 m. The maximum possible value of ΔUT1 is 2.72 ms, which can affect the space-fixed position vector of a tracking station by about 1.3 m. These indirect effects of solid Earth tides are the same order of magnitude as the direct effects. From Eq. (5–42), the radial solid Earth tide varies from about +32 cm to –16 cm.

From Yoder *et al.* (1981), short-period, semi-diurnal, and diurnal ocean tides can cause changes in *C* which produce 0.02 to 0.07 ms semi-diurnal and diurnal UT1 variations. The error in the space-fixed position vector of a tracking station due to these neglected terms of UT1 is about 1 to 3 cm. It will be seen in Subsection 5.3.3.2 that the computed value of ΔUT1 is proportional to the coefficient $k/C$ whose estimated value is 0.94 ± 0.04. The 4% uncertainty in this coefficient can produce errors in the space-fixed position vector of a tracking station of up to 2 cm due to a single term of ΔUT1 and up to 5 cm due to all of the terms.

#### 5.3.3.2 Algorithm for Computing the Short-Period Terms of UT1

Since angular momentum is conserved, the change in Universal Time UT1 due to a change $\delta C$ in the Earth's polar moment of inertia $C$ is given by the second form of Eq. (2) of Yoder *et al.* (1981). The change $\delta C$ (normalized) due to long-period lunar or solar solid Earth tides is given by Eq. (3). This equation is consistent with Eq. (2.154) of Melchior (1966) for $\delta C/C$. Eq. (3) of Yoder *et al.* (1981) gives $\delta C$ as a function of the distance to and the declination of the Moon or the Sun. Eq. (3) is converted to a function of the ecliptic longitude and latitude of the tide-raising body (the Moon or the Sun) and the obliquity of the ecliptic. They list a reference that presumably shows how this equation is expanded. The final expression for the sum ΔUT1 of the 41 short-period terms of UT1 has the form:

$$\Delta \mathrm{UT1} = -\left(\frac{k}{C}\right)\sum_{i=1}^{41} A_i \sin\left(c_{l_i} l + c_{l'_i} l' + c_{F_i} F + c_{D_i} D + c_{\Omega_i} \Omega\right) \qquad \mathrm{s} \qquad (5\text{–}139)$$

where the angles $l$, $l'$, $F$, $D$, and $\Omega$ are the fundamental arguments of the nutation series. They are calculated from Eqs. (5–65) and (5–66) as a function of coordinate time ET of the Solar-System barycentric or local geocentric space-time frame of reference. The positive or negative integer multipliers $c_{l_i}$, $c_{l'_i}$, $c_{F_i}$, $c_{D_i}$, and $c_{\Omega_i}$ of these arguments for each term $i$ of ΔUT1 along with the amplitude $A_i$ of each term are given in Table 5–2. This table is the first part of Table 1 of Yoder *et al.* (1981), which applies for the 41 short-period terms of UT1, which have periods between 5 and 35 days. The second part of Table 1 of Yoder *et al.* (1981) applies for the 21 long-period terms of UT1, which have periods between 91 and 6791 days. Eq. (5–139) contains a minus sign because the data in Table 1 of Yoder *et al.* (1981) applies for − ΔUT1. Their table lists the amplitude $A_i$ for term 22 as

50 x $10^{-7}$ s. However, according to J. G. Williams (personal communication), $A_i$ for term 22 should be –50 x $10^{-7}$ s, which is shown in Table 5–2.

**Table 5–2**

**Short-Period Terms of UT1**

| Term $i$ | Period days | Coefficients of Nutation Angles in Argument | | | | | Amplitude $A_i$ |
|---|---|---|---|---|---|---|---|
| | | $c_{l_i}$ | $c_{l'_i}$ | $c_{F_i}$ | $c_{D_i}$ | $c_{\Omega_i}$ | $10^{-7}$ s |
| 1 | 5.64 | 1 | 0 | 2 | 2 | 2 | 25 |
| 2 | 6.85 | 2 | 0 | 2 | 0 | 1 | 43 |
| 3 | 6.86 | 2 | 0 | 2 | 0 | 2 | 105 |
| 4 | 7.09 | 0 | 0 | 2 | 2 | 1 | 54 |
| 5 | 7.10 | 0 | 0 | 2 | 2 | 2 | 131 |
| 6 | 9.11 | 1 | 0 | 2 | 0 | 0 | 41 |
| 7 | 9.12 | 1 | 0 | 2 | 0 | 1 | 437 |
| 8 | 9.13 | 1 | 0 | 2 | 0 | 2 | 1056 |
| 9 | 9.18 | 3 | 0 | 0 | 0 | 0 | 19 |
| 10 | 9.54 | –1 | 0 | 2 | 2 | 1 | 87 |
| 11 | 9.56 | –1 | 0 | 2 | 2 | 2 | 210 |
| 12 | 9.61 | 1 | 0 | 0 | 2 | 0 | 81 |
| 13 | 12.81 | 2 | 0 | 2 | –2 | 2 | –23 |
| 14 | 13.17 | 0 | 1 | 2 | 0 | 2 | –27 |
| 15 | 13.61 | 0 | 0 | 2 | 0 | 0 | 318 |
| 16 | 13.63 | 0 | 0 | 2 | 0 | 1 | 3413 |
| 17 | 13.66 | 0 | 0 | 2 | 0 | 2 | 8252 |
| 18 | 13.75 | 2 | 0 | 0 | 0 | –1 | –23 |
| 19 | 13.78 | 2 | 0 | 0 | 0 | 0 | 360 |
| 20 | 13.81 | 2 | 0 | 0 | 0 | 1 | –19 |
| 21 | 14.19 | 0 | –1 | 2 | 0 | 2 | 26 |
| 22 | 14.73 | 0 | 0 | 0 | 2 | –1 | –50 |
| 23 | 14.77 | 0 | 0 | 0 | 2 | 0 | 781 |
| 24 | 14.80 | 0 | 0 | 0 | 2 | 1 | 56 |
| 25 | 15.39 | 0 | –1 | 0 | 2 | 0 | 54 |
| 26 | 23.86 | 1 | 0 | 2 | –2 | 1 | –53 |
| 27 | 23.94 | 1 | 0 | 2 | –2 | 2 | –107 |
| 28 | 25.62 | 1 | 1 | 0 | 0 | 0 | –42 |
| 29 | 26.88 | –1 | 0 | 2 | 0 | 0 | –50 |
| 30 | 26.98 | –1 | 0 | 2 | 0 | 1 | –188 |
| 31 | 27.09 | –1 | 0 | 2 | 0 | 2 | –463 |
| 32 | 27.44 | 1 | 0 | 0 | 0 | –1 | –568 |
| 33 | 27.56 | 1 | 0 | 0 | 0 | 0 | 8788 |
| 34 | 27.67 | 1 | 0 | 0 | 0 | 1 | –579 |
| 35 | 29.53 | 0 | 0 | 0 | 1 | 0 | –50 |
| 36 | 29.80 | 1 | –1 | 0 | 0 | 0 | 59 |
| 37 | 31.66 | –1 | 0 | 0 | 2 | –1 | –125 |
| 38 | 31.81 | –1 | 0 | 0 | 2 | 0 | 1940 |
| 39 | 31.96 | –1 | 0 | 0 | 2 | 1 | –140 |
| 40 | 32.61 | 1 | 0 | –2 | 2 | –1 | –19 |
| 41 | 34.85 | –1 | –1 | 0 | 2 | 0 | 91 |

From Yoder *et al.* (1981), the variations in the rotation rate of the Earth's fluid core are decoupled from those of the mantle. Hence, in Eq. (5–139), $k$ is the effective value of the Love number that causes the tidal variation in the polar moment of inertia of the coupled mantle and oceans, and $C$ is the dimensionless polar moment of inertia of these coupled units. The value of $k$ is the Earth's bulk Love number $k_2$ = 0.301 minus 0.064 due to decoupling of the fluid core plus 0.040 due to ocean tides. The estimate of the coefficient $k/C$, which is computed from Eqs. (24) and (28) of Yoder *et al.* (1981), is:

$$\left(\frac{k}{C}\right) = 0.94 \pm 0.04 \tag{5–140}$$

where the 4% uncertainty consists of approximately equal terms due to ocean tide and fluid core uncertainties.

### 5.3.4 PRECESSION MATRIX

In Eq. (5–115) or (5–116), the precession matrix $A$ rotates from coordinates referred to the mean Earth equator and equinox of J2000 (specifically, the radio frame) to coordinates referred to the mean Earth equator and equinox of date. Note that the (mean or true) vernal equinox of date is the ascending node of the ecliptic (the mean orbit plane of the Earth) of date on the (mean or true) Earth equator of date. The definition of the autumnal equinox is obtained from the definition of the vernal equinox by replacing the ascending node of the ecliptic with the descending node. The precession matrix $A$ is currently computed as the following product of three coordinate system rotations:

$$A = R_z(\Delta + \pi)\, R_x\left(\tfrac{\pi}{2} - \delta\right) R_z\left(\alpha + \tfrac{\pi}{2}\right) \tag{5–141}$$

where the coordinate system rotation matrices are given by Eqs. (5–16) to (5–18). The angles $\alpha$ and $\delta$ are the right ascension and declination of the Earth's mean north pole of date relative to the mean Earth equator and equinox of J2000. The angle $\Delta$ is the angle along the mean Earth equator of date from its ascending node on the mean Earth equator of J2000 to the autumnal equinox. Adding $\pi$ to $\Delta$ takes you from the autumnal equinox to the vernal equinox. The angles $\alpha$, $\delta$, and $\Delta$ can be calculated from the equatorial precession angles $\zeta_A$, $z_A$, and $\theta_A$:

$$\begin{aligned} \alpha &= -\zeta_A \\ \delta &= \tfrac{\pi}{2} - \theta_A \qquad \text{rad} \\ \Delta &= \tfrac{\pi}{2} - z_A \end{aligned} \tag{5–142}$$

The equatorial precession angles are given by equations in Table 5 of Lieske *et al*. (1977) or by Eqs. (7) of Lieske (1979). We want these angles to be expressed as polynomials in Julian centuries of coordinate time ET past J2000.0. This is the variable $T$ given by Eq. (5–65). The desired expressions are obtained by setting $T = 0$ in the referenced equations of Lieske. The remaining variable $t$ in these equations is our variable $T$:

$$\begin{aligned} \zeta_A &= 2306''.2181T + 0''.30188T^2 + 0''.017998T^3 \\ z_A &= 2306''.2181T + 1''.09468T^2 + 0''.018203T^3 \\ \theta_A &= 2004''.3109T - 0''.42665T^2 - 0''.041833T^3 \end{aligned} \tag{5–143}$$

These angles can be converted from arcseconds to radians by dividing by 206,264.806,247,096. The geometry used in Eqs. (5–141) and (5–142) is shown in Fig. 1 of Lieske *et al*. (1977) and Lieske (1979).

The precession matrix given by Eq. (5–141) can be simplified. First, substitute $\alpha$, $\delta$, and $\Delta$ from Eqs. (5–142) into (5–141):

$$A = R_z\left(-z_A - \tfrac{\pi}{2}\right) R_x\left(\theta_A\right) R_z\left(\tfrac{\pi}{2} - \zeta_A\right) \tag{5–144}$$

which is the same as:

$$A = R_z\left(-z_A\right) R_z\left(-\tfrac{\pi}{2}\right) R_x\left(\theta_A\right) R_z\left(\tfrac{\pi}{2}\right) R_z\left(-\zeta_A\right) \tag{5–145}$$

Using Eqs. (5–16) to (5–18),

$$R_z\left(-\tfrac{\pi}{2}\right) R_x\left(\theta_A\right) R_z\left(\tfrac{\pi}{2}\right) = R_y\left(\theta_A\right) \tag{5–146}$$

which is obvious from Fig. 1 of Lieske *et al*. (1977) and Lieske (1979). Substituting Eq. (5–146) into (5–145) gives:

$$A = R_z\left(-z_A\right) R_y\left(\theta_A\right) R_z\left(-\zeta_A\right) \tag{5–147}$$

which is also obvious from Fig. 1 of Lieske *et al*. (1977) and Lieske (1979). Lieske (1979) gives two equivalent expressions for the precession matrix $A$ in the unnumbered equation after Eq. (5). The first expression is Eq. (5–144) and the second expression is Eq. (5–147).

The precession matrix $A$ is currently computed from Eq. (5–144) and Eqs. (5–143). However, it would be simpler to calculate $A$ from Eq. (5–147) and Eqs. (5–143). Also, the use of these equations would reduce the roundoff errors in the computed precession matrix.

From Eq. (5–144), the derivative of the precession matrix $A$ with respect to coordinate time ET is given by:

$$\begin{aligned}\dot{A} = &- \frac{dR_z\left(-z_A - \frac{\pi}{2}\right)}{d\left(-z_A - \frac{\pi}{2}\right)} R_x\left(\theta_A\right) R_z\left(\tfrac{\pi}{2} - \zeta_A\right) \dot{z}_A \\ &+ R_z\left(-z_A - \tfrac{\pi}{2}\right) \frac{dR_x\left(\theta_A\right)}{d\left(\theta_A\right)} R_z\left(\tfrac{\pi}{2} - \zeta_A\right) \dot{\theta}_A \qquad \text{rad/s} \\ &- R_z\left(-z_A - \tfrac{\pi}{2}\right) R_x\left(\theta_A\right) \frac{dR_z\left(\frac{\pi}{2} - \zeta_A\right)}{d\left(\frac{\pi}{2} - \zeta_A\right)} \dot{\zeta}_A\end{aligned} \tag{5–148}$$

where the rotation matrices and their derivatives with respect to the rotation angles are given by Eqs. (5–16) to (5–18). The equatorial precession angles are computed from Eqs. (5–143). These equations and the equation for the mean obliquity of the ecliptic ($\bar{\varepsilon}$) (which will be used in the next section) have the form:

$$\zeta_A,\ z_A,\ \theta_A,\ \bar{\varepsilon} = a + bT + cT^2 + dT^3 \qquad \text{arcseconds} \qquad (5\text{–}149)$$

where $T$ is given by Eq. (5–65) and $a$ is zero for the three equatorial precession angles. The time derivatives of these angles in radians per second of coordinate time ET are:

$$\dot{\zeta}_A,\ \dot{z}_A,\ \dot{\theta}_A,\ \dot{\bar{\varepsilon}} = \frac{b + 2cT + 3dT^2}{206{,}264.806{,}247{,}096 \times 86400 \times 36525} \qquad \text{rad/s} \qquad (5\text{–}150)$$

If the precession matrix $A$ was computed from Eq. (5–147) instead of Eq. (5–144), its time derivative $\dot{A}$ would be computed from:

$$\begin{aligned}\dot{A} = &- \frac{dR_z(-z_A)}{d(-z_A)} R_y(\theta_A) R_z(-\zeta_A) \dot{z}_A \\ &+ R_z(-z_A) \frac{dR_y(\theta_A)}{d(\theta_A)} R_z(-\zeta_A) \dot{\theta}_A \qquad \text{rad/s} \qquad (5\text{–}151) \\ &- R_z(-z_A) R_y(\theta_A) \frac{dR_z(-\zeta_A)}{d(-\zeta_A)} \dot{\zeta}_A\end{aligned}$$

### 5.3.5 NUTATION MATRIX

In Eq. (5–115) or (5–116), the nutation matrix $N$ rotates from coordinates referred to the mean Earth equator and equinox of date to coordinates referred to the true Earth equator and equinox of date. The nutation matrix $N$ is computed from the following sequence of three coordinate system rotations:

$$N = R_x(-\bar{\varepsilon} - \Delta\varepsilon) R_z(-\Delta\psi) R_x(\bar{\varepsilon}) \qquad (5\text{–}152)$$

where the coordinate system rotation matrices are given by Eqs. (5–16) to (5–18). The mean obliquity of the ecliptic $\bar{\varepsilon}$ is the inclination of the ecliptic (the mean orbit plane of the Earth) of date to the mean Earth equator of date. It is given by equations in Table 5 of Lieske *et al.* (1977). We want it to be expressed as a polynomial in Julian centuries of coordinate time ET past J2000.0, which is the variable $T$ given by Eq. (5–65). The desired expression is obtained by setting the variable $T = 0$ in the equations for $\varepsilon_A$ and $\bar{\varepsilon}_A$ in Table 5 of Lieske *et al*. (1977) and denoting their variable $t$ as our variable $T$:

$$\bar{\varepsilon} = 84{,}381''.448 - 46''.8150T - 0''.00059T^2 + 0''.001813T^3 \qquad (5\text{–}153)$$

This angle can be converted from arcseconds to radians by dividing by 206,264.806,247,096. The coordinate system rotations in Eq. (5–152) are based upon the geometry in Fig. 3.222.1 on p. 115 of *Explanatory Supplement* (1992). Eq. (3.222–3) of this reference is the same as Eq. (5–152). In the former equation, the true obliquity of the ecliptic $\varepsilon$ is the inclination of the ecliptic of date to the true Earth equator of date. It is the sum of the mean obliquity $\bar{\varepsilon}$ and the nutation in obliquity $\Delta\varepsilon$:

$$\varepsilon = \bar{\varepsilon} + \Delta\varepsilon \qquad \text{rad} \qquad (5\text{–}154)$$

From the referenced figure, the nutation in longitude $\Delta\psi$ is the celestial longitude (measured in the ecliptic) of the mean equinox of date measured from the true equinox of date. The nutation in longitude $\Delta\psi$ and the nutation in obliquity $\Delta\varepsilon$ in radians and their time derivatives $(\Delta\psi)^{\cdot}$ and $(\Delta\varepsilon)^{\cdot}$ in radians per second are obtained as described in Section 5.3.2. These quantities are the sum of the quantities (5–136) obtained from the 1980 IAU Theory of Nutation (Seidelmann, 1982) plus the corrections (5–137) obtained from the EOP file. We use the notation of the former quantities to denote the sum of (5–136) and (5–137), which contains the corrected nutation angles and their time derivatives.

From Eq. (5–152), the derivative of the nutation matrix $N$ with respect to coordinate time ET is given by:

$$\dot{N} = -\frac{dR_x(-\bar{\varepsilon}-\Delta\varepsilon)}{d(-\bar{\varepsilon}-\Delta\varepsilon)} R_z(-\Delta\psi)\, R_x(\bar{\varepsilon})\left[\dot{\bar{\varepsilon}} + (\Delta\varepsilon)^{\cdot}\right]$$

$$- R_x(-\bar{\varepsilon}-\Delta\varepsilon)\frac{dR_z(-\Delta\psi)}{d(-\Delta\psi)} R_x(\bar{\varepsilon})\,(\Delta\psi)^{\cdot} \qquad \text{rad/s} \qquad (5\text{–}155)$$

$$+ R_x(-\bar{\varepsilon}-\Delta\varepsilon)\, R_z(-\Delta\psi)\frac{dR_x(\bar{\varepsilon})}{d(\bar{\varepsilon})}\dot{\bar{\varepsilon}}$$

where the rotation matrices and their derivatives with respect to the rotation angles are given by Eqs. (5–16) to (5–18). The time derivative $\dot{\bar{\varepsilon}}$ of the mean obliquity of the ecliptic is calculated from Eqs. (5–153), (5–149), and (5–150).

### 5.3.6 ROTATION MATRIX THROUGH TRUE SIDEREAL TIME

In Eq. (5–115) or (5–116), the matrix $B$ rotates from space-fixed coordinates referred to the true Earth equator and equinox of date to Earth-fixed coordinates referred to the true pole, prime meridian, and equator of date. Subsection 5.3.6.1 gives the formulas for $B$, its time derivative $\dot{B}$, its second time derivative $\ddot{B}$, and the partial derivative of $B$ with respect to Universal Time UT1. These quantities are a function of true sidereal time $\theta$, its time derivative $\dot{\theta}$, and the partial derivative of $\theta$ with respect to UT1. The formulation for calculating these three quantities is given in Subsection 5.3.6.2. The matrix $\ddot{B}$ is used to calculate $\ddot{T}_E$ given by Eq. (5–129). Subsection 5.3.6.1 gives a simple algorithm for calculating $\ddot{T}_E$. Calculation of true sidereal time $\theta$ requires that the time argument, which is coordinate time ET, be transformed to Universal Time UT1 using the complete expression for ET − TAI in the Solar-System barycentric frame. Evaluation of this time difference requires the geocentric space-fixed position vector of the tracking station, which can be calculated from the approximate algorithm given in Subsection 5.3.6.3.

#### 5.3.6.1 Rotation Matrix *B*, its Time Derivatives, and Partial Derivative With Respect to Universal Time UT1

The matrix $B$ rotates from space-fixed coordinates referred to the true Earth equator and equinox of date to Earth-fixed coordinates referred to the true pole, prime meridian, and equator of date. It is a rotation about the z axis through true sidereal time $\theta$:

$$B = R_z(\theta) \tag{5–156}$$

where the coordinate system rotation matrix is given by Eq. (5–18). True sidereal time $\theta$ is the Greenwich hour angle of the Earth's true vernal equinox of date. It is measured westward from the true prime (*i.e.*, 0°) meridian of date about the true pole of date to the true vernal equinox of date.

The derivative of the rotation matrix $B$ with respect to coordinate time ET is given by:

$$\dot{B} = \frac{dR_z(\theta)}{d\theta}\,\dot{\theta} \qquad \text{rad/s} \tag{5–157}$$

where the derivative of the coordinate system rotation matrix with respect to the coordinate system rotation angle is given by Eq. (5–18). The sidereal rate $\dot{\theta}$ is the derivative of true sidereal time $\theta$ with respect to coordinate time ET.

The second time derivative of the rotation matrix $B$ with respect to coordinate time ET is given to sufficient accuracy by:

$$\ddot{B} = -\left[R_z(\theta)\right]^* \dot{\theta}^2 \qquad \text{rad/s}^2 \tag{5–158}$$

where the $*$ indicates that the (3,3) element of the rotation matrix given by Eq. (5–18) is changed from 1 to 0. The desired expression for the second time derivative of $T_E$ can be obtained by substituting Eq. (5–158) into Eq. (5–129). However, this process will be accomplished in two steps. First, substitute Eq. (5–158) without the superscript $*$, Eq. (5–156), and Eq. (5–115) into Eq. (5–129), which gives:

$$\ddot{T}_E = -\,T_E\,\dot{\theta}^2 \qquad \text{rad/s}^2 \tag{5–159}$$

The second step is to correct the calculation of $T_{\mathrm{E}}$ from Eq. (5–115) by setting the (3,3) element of $B$ given by Eq. (5–156) and Eq. (5–18) to zero. In Eq. (5–115), this change zeroes out row three inside of the parentheses and zeroes out column three after taking the transpose. Hence, $\ddot{T}_{\mathrm{E}}$ can be calculated by evaluating Eq. (5–159) and then setting column three of this 3 x 3 matrix to zero.

From Eq. (5–156), the partial derivative of the rotation matrix $B$ with respect to Universal Time UT1 is given by:

$$\frac{\partial B}{\partial \mathrm{UT1}} = \frac{dR_{\mathrm{z}}(\theta)}{d\theta}\frac{\partial\theta}{\partial \mathrm{UT1}} \qquad \mathrm{rad/s} \qquad (5\text{–}160)$$

where the derivative of the rotation matrix with respect to the rotation angle is given by Eq. (5–18).

### 5.3.6.2 Sidereal Time, Its Time Derivative, and Partial Derivative With Respect to Universal Time UT1

True sidereal time $\theta$ is calculated as the sum of mean sidereal time $\theta_{\mathrm{M}}$ plus the equation of the equinoxes $\Delta\theta$:

$$\theta = \theta_{\mathrm{M}} + \Delta\theta \qquad \mathrm{rad} \qquad (5\text{–}161)$$

Mean sidereal time $\theta_{\mathrm{M}}$ is the Greenwich hour angle of the Earth's mean vernal equinox of date. It is measured westward from the true prime meridian of date about the true pole of date to the meridian that contains the mean vernal equinox of date. Subsection 5.3.6.2.1 develops the equations for calculating mean sidereal time $\theta_{\mathrm{M}}$, its time derivative $\dot{\theta}_{\mathrm{M}}$ with respect to coordinate time ET, and its approximate derivative with respect to Universal Time UT1. Subsection 5.3.6.2.2 gives the existing formulation for calculating the equation of the equinoxes $\Delta\theta$ and its time derivative $(\Delta\theta)^{\cdot}$ with respect to coordinate time ET. Subsection 5.3.6.2.3 gives the proposed International Earth Rotation Service (IERS) equation for $\Delta\theta$ and its time derivative $(\Delta\theta)^{\cdot}$.

True sidereal time $\theta$ is actually calculated from the following version of Eq. (5–161):

$$\theta = \left[ \left( \theta_M{}^r + \Delta\theta^r \right)_{\substack{\text{fractional}\\ \text{part}}} \right] 2\pi \qquad \text{rad} \qquad (5\text{–}162)$$

where the superscript r indicates that the quantity has the units of revolutions, where one revolution of the quantity is $2\pi$ radians or $1296000''$. The subscript "fractional part" indicates that true sidereal time $\theta$ in revolutions is computed modulo 1 revolution. That is, the integral number of revolutions of $\theta$ are discarded leaving $\theta$ as a fraction of one revolution. Multiplying by $2\pi$ converts $\theta$ to radians. If sidereal time $\theta$ is calculated one Julian century before of after J2000, 36625 revolutions of sidereal time will be discarded. Hence, five significant digits of $\theta$ will be lost.

From Eq. (5–161), the derivative of true sidereal time $\theta$ with respect to coordinate time ET is given by:

$$\dot{\theta} = \dot{\theta}_M + \left( \Delta\theta \right)^{\cdot} \qquad \text{rad/s} \qquad (5\text{–}163)$$

In Eq. (5–161), mean sidereal time $\theta_M$ is a function of Universal Time UT1 and the equation of the equinoxes $\Delta\theta$ is a function of coordinate time ET. Hence,

$$\frac{\partial\theta}{\partial \text{UT1}} = \frac{d\theta_M}{d\,\text{UT1}} \qquad \text{rad/s} \qquad (5\text{–}164)$$

5.3.6.2.1 Mean Sidereal Time and Its Time Derivatives

From p. S13 of *Supplement To The Astronomical Almanac 1984*, the expression for mean sidereal time $\theta_M$ at $0^h$ UT1 is given by:

$$\begin{aligned}\theta_M\left(0^h\,\text{UT1}\right) = {} & 24{,}110^s.548{,}41 + 8{,}640{,}184^s.812{,}866\, T_U \\ & + 0^s.093{,}104\, T_U{}^2 - 6^s.2\times 10^{-6}\, T_U{}^3\end{aligned} \qquad (5\text{–}165)$$

where

$T_U$ = Julian centuries of 36525 days of 86400 s of Universal Time UT1 elapsed since January 1, 2000, $12^h$ UT1 (J2000.0; JD 245,1545.0)

$$= \frac{\mathrm{UT1}}{86400 \times 36525} \tag{5–166}$$

where

UT1 = seconds of Universal Time UT1 elapsed since January 1, 2000, $12^h$ UT1.

Note that UT1 is an elapsed interval of UT1 time. UT1 time, which is measured in seconds past the start of the day, is equal to the interval UT1, defined above, plus $12^h$. The interval UT1 used in Eq. (5–166) is obtained by transforming coordinate time ET (measured in seconds past January 1, 2000, $12^h$ ET) as described in detail in Section 5.3.2, item 5.

We need to convert Eq. (5–165) to a general expression for mean sidereal time $\theta_M$ at the current value of UT1. This can be done by using the artifice of the fictitious mean Sun which moves in the equatorial plane at a nearly constant rate. Universal Time UT1 is equal to the hour angle of the fictitious mean Sun (HAMS) plus 12 hours:

$$\mathrm{UT1} = \mathrm{HAMS} + 12^h \tag{5–167}$$

Also, mean sidereal time is equal to the hour angle of the fictitious mean Sun plus the right ascension of the fictitious mean Sun:

$$\theta_M = \mathrm{HAMS} + \mathrm{RAMS} \tag{5–168}$$

Substituting HAMS from (5–167) into (5–168) gives:

$$\theta_M = \mathrm{UT1} + \left(\mathrm{RAMS} - 12^h\right) \tag{5–169}$$

At $0^{\mathrm{h}}$ UT1,

$$\theta_{\mathrm{M}}\left(0^{\mathrm{h}}\ \mathrm{UT1}\right) = \left(\mathrm{RAMS} - 12^{\mathrm{h}}\right) \tag{5–170}$$

Substituting the right-hand side of (5–170) into (5–169) gives the desired expression for mean sidereal time $\theta_{\mathrm{M}}$:

$$\theta_{\mathrm{M}} = \mathrm{UT1} + \theta_{\mathrm{M}}\left(0^{\mathrm{h}}\ \mathrm{UT1}\right) \tag{5–171}$$

where the second term on the right-hand side is Eq. (5–165) evaluated at the current value of $T_{\mathrm{U}}$, not at $0^{\mathrm{h}}$ UT1 time. The first term on the right-hand side is UT1 time, which is the interval UT1 in Eq. (5–166) plus $12^{\mathrm{h}}$. From Eq. (5–166), the interval UT1 can be expressed as:

$$\mathrm{UT1} = 3{,}155{,}760{,}000^{\mathrm{s}} \times T_{\mathrm{U}} \tag{5–172}$$

Hence, from Eq. (5–171) and the explanation following it, the expression for mean sidereal time $\theta_{\mathrm{M}}$ is Eq. (5–165) plus $12^{\mathrm{h}} = 43200^{\mathrm{s}}$ plus the interval UT1 given by Eq. (5–172):

$$\begin{aligned}\theta_{\mathrm{M}} &= 67{,}310^{\mathrm{s}}.548{,}41 + \left(3{,}155{,}760{,}000^{\mathrm{s}}. + 8{,}640{,}184^{\mathrm{s}}.812{,}866\right) T_{\mathrm{U}} \\ &\quad + 0^{\mathrm{s}}.093{,}104\ T_{\mathrm{U}}^{\ 2} - 6^{\mathrm{s}}.2 \times 10^{-6}\ T_{\mathrm{U}}^{\ 3}\end{aligned} \tag{5–173}$$

which is the equation for GMST at the bottom of p. S15 of *Supplement To The Astronomical Almanac 1984*. Eq. (5–162) requires $\theta_{\mathrm{M}}$ in revolutions, which is given by:

$$\theta_{\mathrm{M}}^{\ \mathrm{r}} = \frac{J + K\,T_{\mathrm{U}} + L\,T_{\mathrm{U}}^{\ 2} + M\,T_{\mathrm{U}}^{\ 3}}{86400} \qquad \mathrm{rev} \tag{5–174}$$

where

$J$ = $67{,}310^{s}.548{,}41$

$K$ = $3{,}164{,}400{,}184^{s}.812{,}866$

$L$ = $0^{s}.093{,}104$

$M$ = $-\,6^{s}.2 \times 10^{-6}$

From Eq. (5–174) and (5–166), the derivative of mean sidereal time $\theta_{\mathrm{M}}$ with respect to Universal Time UT1 in radians per second is given by:

$$\frac{d\theta_{\mathrm{M}}}{d\,\mathrm{UT1}} = \frac{K + 2LT_{\mathrm{U}} + 3MT_{\mathrm{U}}^{2}}{(86400)^{2} \times 36525}\,2\pi \qquad \mathrm{rad/s} \qquad (5\text{–}175)$$

An approximate value of this derivative, required for use in Eqs. (5–164), (5–160), and (5–135) is given by:

$$\frac{d\theta_{\mathrm{M}}}{d\,\mathrm{UT1}} = \frac{2\pi\,K}{(86400)^{2} \times 36525} = 0.729{,}211{,}59 \times 10^{-4} \quad \mathrm{rad/s} \qquad (5\text{–}176)$$

The derivative of $\theta_{\mathrm{M}}$ with respect to coordinate time ET is given by:

$$\dot{\theta}_{\mathrm{M}} = \frac{d\theta_{\mathrm{M}}}{d\,\mathrm{UT1}}\,\frac{d\,\mathrm{UT1}}{d\,\mathrm{ET}} \qquad \mathrm{rad/s} \qquad (5\text{–}177)$$

The transformation of coordinate time ET to Universal Time UT1, which is described in Section 5.3.2, item 5, is given by:

$$\mathrm{UT1} = \mathrm{ET} - (\mathrm{ET} - \mathrm{TAI}) - (\mathrm{TAI} - \mathrm{UT1}) + \Delta\mathrm{UT1} \qquad \mathrm{s} \qquad (5\text{–}178)$$

where I have assumed that the TP array or the EOP file contains regularized UT1. The derivative of UT1 with respect to ET is given by:

$$\frac{d\,\mathrm{UT1}}{d\,\mathrm{ET}} = 1 - (\mathrm{ET} - \mathrm{TAI})^{\cdot} - (\mathrm{TAI} - \mathrm{UT1})^{\cdot} + (\Delta\mathrm{UT1})^{\cdot} \qquad \mathrm{s/s} \qquad (5\text{–}179)$$

Since the computed values of observed quantities are computed from position coordinates or differenced position coordinates, accurate velocities are not required in program Regres. Hence, on the right-hand side of (5–179), we only need to keep the largest time derivative, which is $(\mathrm{TAI}-\mathrm{UT1})^{\cdot}$. It can be as large as 0.4 x $10^{-7}$ s/s. Substituting this approximation to Eq. (5–179) and Eq. (5–175) into Eq. (5–177) gives:

$$\dot{\theta}_{\mathrm{M}} = \frac{K + 2LT_{\mathrm{U}} + 3MT_{\mathrm{U}}^{2}}{(86400)^{2} \times 36525}\left[1-(\mathrm{TAI}-\mathrm{UT1})^{\cdot}\right] 2\pi \qquad \mathrm{rad/s} \qquad (5\text{–}180)$$

This equation is used in Eq. (5–163).

5.3.6.2.2 Existing Formulation for the Equation of the Equinoxes

The existing expression for the equation of the equinoxes is:

$$\Delta\theta = \Delta\psi \cos(\bar{\varepsilon} + \Delta\varepsilon) \qquad \mathrm{rad} \qquad (5\text{–}181)$$

where the nutation in longitude $\Delta\psi$ and the nutation in obliquity $\Delta\varepsilon$ are obtained as described in Section 5.3.2 and include the corrections obtained from the EOP file. The mean obliquity of the ecliptic $\bar{\varepsilon}$ is calculated from Eq. (5–153) and then converted to radians. Eq. (5–181) is based upon the geometry shown in Fig. 3.222.1 on p. 115 of the *Explanatory Supplement* (1992). Eq. (5–162) requires $\Delta\theta$ in revolutions, which is given by:

$$\Delta\theta^{\mathrm{r}} = \frac{\Delta\psi \cos(\bar{\varepsilon} + \Delta\varepsilon)}{2\pi} \qquad \mathrm{rev} \qquad (5\text{–}182)$$

Eq. (5–163) uses the derivative of $\Delta\theta$ with respect to coordinate time ET in radians per second. From (5–181), it is given by:

$$\begin{aligned}(\Delta\theta)^{\cdot} &= (\Delta\psi)^{\cdot}\cos(\bar{\varepsilon} + \Delta\varepsilon) \\ &\quad - (\Delta\psi)\sin(\bar{\varepsilon} + \Delta\varepsilon)\left[\dot{\bar{\varepsilon}} + (\Delta\varepsilon)^{\cdot}\right]\end{aligned} \qquad \mathrm{rad/s} \qquad (5\text{–}183)$$

where $(\Delta\psi)^{\cdot}$ and $(\Delta\varepsilon)^{\cdot}$ are obtained as described in Section 5.3.2 and $\dot{\bar{\varepsilon}}$ is calculated from Eqs. (5–153), (5–149), and (5–150).

5.3.6.2.3 Proposed Formulation for the Equation of the Equinoxes

From page 30 of International Earth Rotation Service (1992) and pages 21–22 of International Earth Rotation Service (1996), the proposed expression for the equation of the equinoxes, which should be used starting on January 1, 1997, is:

$$\Delta\theta = \Delta\psi \cos\bar{\varepsilon} + 0''.00264 \sin\Omega + 0''.000063 \sin 2\Omega \tag{5–184}$$

where $\Omega$ is the longitude of the mean ascending node of the lunar orbit on the ecliptic. It is defined by Eq. (5–64) and calculated from Eq. (5–66). Eq. (5–184) is Eq. (A2–35) of Aoki and Kinoshita (1983).

The existing expression for the equation of the equinoxes is given by Eq. (5–181). Expanding this equation and retaining all terms to the second order in the nutations gives:

$$\Delta\theta = \Delta\psi \cos\bar{\varepsilon} - \Delta\psi\left(\sin\bar{\varepsilon}\right)\Delta\varepsilon \tag{5–185}$$

The first term of this expression is the first term of Eq. (5–184). Differentiating the second term with respect to time gives:

$$-\left(\Delta\psi\right)^{\cdot}\left(\sin\bar{\varepsilon}\right)\Delta\varepsilon - \Delta\psi\left(\sin\bar{\varepsilon}\right)\left(\Delta\varepsilon\right)^{\cdot} \tag{5–186}$$

where the derivative of $\sin\bar{\varepsilon}$ has been ignored. If the expression (5–186) were integrated with respect to time, we would obtain the second term of Eq. (5–185). Adding it to the first term of this equation would give the existing expression (5–181) for the equation of the equinoxes. The first term of (5–186) is integrated with respect to time to give a periodic term of the new expression for the equation of the equinoxes. Integration of the second term of (5–186) with respect to time would give another periodic term in the equation of the equinoxes. This term represents a periodic movement of the true meridian containing the mean equinox of date relative to the true equator of date. The periodic movement of this meridian also produces an equal and opposite periodic term in the expression for mean sidereal time. These equal and opposite terms cancel in calculating true sidereal time from Eq. (5–161). Hence, the second term of (5–186) is discarded. Its time integral is not included in the new expression for the equation of the equinoxes.

The accumulated luni-solar precession in right ascension along the true equator of date is given by:

$$\int \dot{\psi} \cos(\bar{\varepsilon} + \Delta\varepsilon)\, dt \tag{5–187}$$

where planetary precession is ignored and $\dot{\psi}$ is the rate of luni-solar precession along the ecliptic. Expanding gives the accumulated luni-solar precession in right ascension, which is included in the precession matrix (5–147), and the following term:

$$-\int \dot{\psi}\, (\sin\bar{\varepsilon})\, \Delta\varepsilon\, dt \tag{5–188}$$

which is a periodic variation in the accumulated precession in right ascension due to the nutation in obliquity $\Delta\varepsilon$.

The new expression for the equation of the equinoxes is given by the first term of Eq. (5–185) plus the time integral of the first term of (5–186) plus the term (5–188):

$$\Delta\theta = \Delta\psi \cos\bar{\varepsilon} - \int \dot{\psi}\, (\sin\bar{\varepsilon})\, \Delta\varepsilon\, dt - \left[ \int (\Delta\psi)^{\cdot} (\sin\bar{\varepsilon})\, \Delta\varepsilon\, dt \right]_{\mathrm{p}} \tag{5–189}$$

where the subscript p indicates that only the periodic terms are retained. This equation is the same as the first three terms of Eq. (A2–33) of Aoki and Kinoshita (1983). The authors state that the remaining terms of this equation are negligible.

Eq. (5–189) can be evaluated by evaluating the nutations in longitude and obliquity from selected terms of the series expressions for these quantities. First, from Table 1 of Seidelmann (1982), evaluate the nutations in longitude and obliquity from term 1 of the series expressions for these angles:

$$\begin{aligned} \Delta\psi &= -\,17''.1996 \sin\Omega \\ \Delta\varepsilon &= \quad 9''.2025 \cos\Omega \end{aligned} \tag{5–190}$$

Substituting these equations, $\bar{\varepsilon}$ obtained from Eq. (5–153) at J2000, and $\dot{\psi}$ obtained from Table 3.211.1 on p. 104 of the *Explanatory Supplement* (1992) into terms 2 and 3 of Eq. (5–189) and using $\dot{\Omega}$ obtained from Eq. (5–66) gives:

$$0''.00265 \sin\Omega \tag{5–191}$$

which is obtained from term 2 of (5–189), and

$$0''.000076 \sin 2\Omega \tag{5–192}$$

which is obtained from term 3 of (5–189). Then, from Table 1 of Seidelmann (1982), evaluate the nutation in obliquity from term 2 of the series expression for this angle:

$$\Delta\varepsilon = -\,0''.0895 \cos 2\Omega \tag{5–193}$$

Substituting this equation into term 2 of Eq. (5–189) gives:

$$-\,0''.000013 \sin 2\Omega \tag{5–194}$$

Evaluating the second term of Eq. (5–189) as the sum of terms (5–191) and (5–194), and the third term as (5–192) gives Eq. (5–184) for the new expression for the equation of the equinoxes, except for a change of 0″.00001 in the coefficient of the $\sin\Omega$ term.

Eq. (5–162) requires $\Delta\theta$ in revolutions, which is given by:

$$\Delta\theta^{\mathrm{r}} = \frac{\Delta\psi \cos\bar{\varepsilon}}{2\pi} + \frac{0''.00264 \sin\Omega + 0''.000063 \sin 2\Omega}{1,296,000} \qquad \text{rev} \tag{5–195}$$

Eq. (5–163) uses the derivative of $\Delta\theta$ with respect to coordinate time ET in radians per second. From Eq. (5–184), it is given by:

$$(\Delta\theta)^{\cdot} = (\Delta\psi)^{\cdot}\cos\bar{\varepsilon} - (\Delta\psi)(\sin\bar{\varepsilon})\,\dot{\bar{\varepsilon}} + \frac{0''.00264\cos\Omega + 2\times 0''.000063\cos 2\Omega}{206{,}264.806{,}247{,}096}\,\dot{\Omega} \qquad \text{rad/s} \tag{5–196}$$

Since $\Omega$ given by Eq. (5–66) and $\bar{\varepsilon}$ given by Eq. (5–153) have the same form, their derivatives with respect to coordinate time ET can be calculated using Eqs. (5–149) and (5–150).

From Eq. (A2–36) of Aoki and Kinoshita (1983), the sum of the secular terms, which were discarded from the third term of Eq. (5–189), is given by:

$$-\,0''.00388\,T \tag{5–197}$$

where $T$ is given by Eq. (5–65). In principle, (5–197) should be added to Eq. (5–173) for mean sidereal time. In practice, this change will not be made, and the neglected term will be absorbed into the "observed" value of Universal Time UT1. After one century, UT1 will change by 2.6 x $10^{-4}$ s. This is quite negligible compared to leap seconds, which occur on the order of once a year.

### 5.3.6.3 Algorithm for Approximate Geocentric Space-Fixed Position Vector of Tracking Station

In Section 5.3.2, Item 5, the time argument in coordinate time ET is transformed to Universal Time UT1 using the complete expression for the time difference ET − TAI in the Solar-System barycentric frame of reference. This expression is Eq. (2–23), which can be evaluated using the very approximate algorithm for the geocentric space-fixed position vector of the tracking station $\mathbf{r}_{\mathrm{A}}^{\mathrm{E}}$, which is given in this section.

True sidereal time $\theta$ is approximated by mean sidereal time $\theta_{\mathrm{M}}$, given by Eq. (5–174). In this equation, the $L$ and $M$ coefficients are ignored, and $T_{\mathrm{U}}$ given by Eq. (5–166) is approximated by $T$ given by Eq. (5–65). Hence,

$$\theta = \left[ \left( \frac{J + K\,T}{86400} \right)_{\substack{\text{fractional}\\ \text{part}}} \right] 2\pi \qquad \text{rad} \qquad (5\text{–}198)$$

and the geocentric space-fixed position vector of the tracking station is given approximately by:

$$\mathbf{r}_{\mathrm{A}}^{\mathrm{E}} = \begin{bmatrix} u\cos(\theta + \lambda) \\ u\sin(\theta + \lambda) \\ v \end{bmatrix} \qquad \text{km} \qquad (5\text{–}199)$$

where $u$, $v$, and $\lambda$ are the input Earth-fixed 1903.0 cylindrical coordinates of the tracking station, uncorrected for polar motion.

The error in $\mathbf{r}_{\mathrm{A}}^{\mathrm{E}}$ calculated from Eqs. (5–198) and (5–199) is less than 300 km. From the fourth term on the right-hand side of Eq. (2–23), the resulting error in TAI and UT1 is less than $10^{-7}$ s. This will produce an error in the space-fixed position vector of the tracking station, calculated from Eq. (5–113) of 0.004 cm, which is negligible.

## 5.4 GEOCENTRIC SPACE-FIXED POSITION, VELOCITY, AND ACCELERATION VECTORS OF TRACKING STATION

### 5.4.1 ROTATION FROM EARTH-FIXED TO SPACE-FIXED COORDINATES

The transformation from the Earth-fixed position vector $\mathbf{r}_{\mathrm{b}}$ of a tracking station on Earth to the corresponding space-fixed position vector $\mathbf{r}_{\mathrm{TS}}^{\mathrm{E}}$ of the tracking station relative to the Earth is given by Eq. (5–113). The variables in this equation are described in the paragraph containing Eq. (5–113).

Calculation of the computed values of observed quantities (*e.g.*, doppler and range observables) requires accurate and precise values of position vectors of the participants (*e.g.*, the spacecraft and the tracking station). Since the computed values of doppler observables are calculated from differenced round-trip light times divided by their time separation, high-accuracy velocity and acceleration vectors are not required in program Regres. The maximum Earth-fixed velocity of the tracking station is about 3 x $10^{-5}$ m/s due to solid Earth tides. This affects the tenth significant digit of the velocity of the tracking station relative to the Solar-System barycenter, which can be ignored. Hence, the geocentric space-fixed velocity and acceleration vectors of the tracking station can be computed from derivatives of Eq. (5–113) with respect to coordinate time ET holding $\mathbf{r}_b$ fixed:

$$\dot{\mathbf{r}}_{TS}^{E} = \dot{T}_E \, \mathbf{r}_b \qquad \text{km/s} \qquad (5\text{–}200)$$

$$\ddot{\mathbf{r}}_{TS}^{E} = \ddot{T}_E \, \mathbf{r}_b \qquad \text{km/s}^2 \qquad (5\text{–}201)$$

where $\dot{T}_E$ is given by Eq. (5–128). The formulations for the time derivatives in this equation are all available within Section 5.3. The second time derivative of $T_E$ is obtained by evaluating Eq. (5–159) and then setting column three of this 3 x 3 matrix to zero.

### 5.4.2 TRANSFORMATION OF GEOCENTRIC SPACE-FIXED POSITION VECTOR FROM LOCAL GEOCENTRIC TO SOLAR-SYSTEM BARYCENTRIC RELATIVISTIC FRAME OF REFERENCE

The geocentric space-fixed position vector of the tracking station calculated from Eq. (5–113) is in the local geocentric space-time frame of reference. If Regres is operating in this frame of reference, no further calculations are required. However, if Regres is operating in the Solar-System barycentric relativistic frame of reference, then this vector must be transformed from the local geocentric to the Solar-System barycentric relativistic frame of reference using Eq. (4–10).

In Eq. (4–10), $\mathbf{r}_{GC}$ is $\mathbf{r}_{TS}^{E}$ calculated from Eq. (5–113). Calculation of the remaining variables in (4–10) is described in the paragraph after Eq. (4–11). In evaluating the gravitational potential $U_{\mathrm{E}}$ at the Earth, the only term that needs to be included is the potential due to the Sun. The constant $\tilde{L}$ is given by (4–17).

## 5.5 PARTIAL DERIVATIVES OF GEOCENTRIC SPACE-FIXED POSITION VECTOR OF TRACKING STATION

This section gives the formulation for calculating partial derivatives of the geocentric space-fixed position vector $\mathbf{r}_{\mathrm{TS}}^{\mathrm{E}}$ of the tracking station with respect to solve-for or consider parameters. These partial derivatives can be used to estimate the values of the parameters (*i.e.*, solve-for parameters) or to consider the uncertainty in the parameters when calculating the covariance matrix for the estimated parameters (*i.e.*, consider parameters). Subsection 5.5.1 gives the partial derivatives for the parameters which affect the Earth-fixed position vector $\mathbf{r}_{\mathrm{b}}$ of the tracking station. The next two Subsections give partials for parameters which affect the Earth-fixed to space-fixed transformation matrix $T_{\mathrm{E}}$. Subsection 5.5.2 gives the partial derivatives for the frame-tie rotation angles $r_{\mathrm{z}}$, $r_{\mathrm{y}}$, and $r_{\mathrm{x}}$. Subsection 5.5.3 gives the partial derivative with respect to Universal Time UT1, which affects mean sidereal time $\theta_{\mathrm{M}}$.

### 5.5.1 PARAMETERS AFFECTING EARTH-FIXED POSITION VECTOR OF TRACKING STATION

From Eq. (5–113), for those parameters $\mathbf{q}$ which affect $\mathbf{r}_{\mathrm{b}}$ and not $T_{\mathrm{E}}$,

$$\frac{\partial \mathbf{r}_{\mathrm{TS}}^{\mathrm{E}}}{\partial \mathbf{q}} = T_{\mathrm{E}} \frac{\partial \mathbf{r}_{\mathrm{b}}}{\partial \mathbf{q}} \tag{5–202}$$

From Eqs. (5–1) and (5–2), the partial derivatives of $\mathbf{r}_{\mathrm{b}}$ with respect to the input 1903.0 cylindrical coordinates $u$, $v$, and $\lambda$ of the tracking station are:

$$\frac{\partial \mathbf{r}_{\mathrm{b}}}{\partial u} = \begin{bmatrix} \cos\lambda \\ \sin\lambda \\ 0 \end{bmatrix} \alpha = \begin{bmatrix} x_{\mathrm{b}_0} \\ y_{\mathrm{b}_0} \\ 0 \end{bmatrix} \frac{\alpha}{u} \tag{5–203}$$

where the components in the second matrix on the right-hand side are those of Eq. (5–2).

$$\frac{\partial \mathbf{r}_{\mathrm{b}}}{\partial v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \alpha \tag{5–204}$$

$$\frac{\partial \mathbf{r}_{\mathrm{b}}}{\partial \lambda} = \begin{bmatrix} -u \sin\lambda \\ u \cos\lambda \\ 0 \end{bmatrix} \alpha = \begin{bmatrix} -y_{\mathrm{b}_0} \\ x_{\mathrm{b}_0} \\ 0 \end{bmatrix} \alpha \tag{5–205}$$

From Eqs. (5–1) and (5–3), the partial derivatives of $\mathbf{r}_{\mathrm{b}}$ with respect to the input 1903.0 spherical coordinates $r$, $\phi$, and $\lambda$ of the tracking station are:

$$\frac{\partial \mathbf{r}_{\mathrm{b}}}{\partial r} = \begin{bmatrix} \cos\phi \cos\lambda \\ \cos\phi \sin\lambda \\ \sin\phi \end{bmatrix} \alpha = \mathbf{r}_{\mathrm{b}_0} \left( \frac{\alpha}{r} \right) \tag{5–206}$$

where $\mathbf{r}_{\mathrm{b}_0}$ is given by Eq. (5–3).

$$\frac{\partial \mathbf{r}_{\mathrm{b}}}{\partial \phi} = \begin{bmatrix} -r \sin\phi \cos\lambda \\ -r \sin\phi \sin\lambda \\ r \cos\phi \end{bmatrix} \alpha \tag{5–207}$$

$$\frac{\partial \mathbf{r}_{\mathrm{b}}}{\partial \lambda} = \begin{bmatrix} -r \cos\phi \sin\lambda \\ r \cos\phi \cos\lambda \\ 0 \end{bmatrix} \alpha = \begin{bmatrix} -y_{\mathrm{b}_0} \\ x_{\mathrm{b}_0} \\ 0 \end{bmatrix} \alpha \tag{5–208}$$

where the components in the second matrix on the right-hand side are those of Eq. (5–3). From Eq. (5–1), the partial derivative of $\mathbf{r}_{\mathrm{b}}$ with respect to the scale factor $\alpha$ is given by:

$$\frac{\partial \mathbf{r}_{\mathrm{b}}}{\partial \alpha} = \mathbf{r}_{\mathrm{b}_0} \tag{5–209}$$

where $\mathbf{r}_{\mathrm{b}_0}$ is given by Eq. (5–2) or (5–3).

From Eqs. (5–1) and (5–12), the partial derivatives of $\mathbf{r}_\mathrm{b}$ with respect to the north ($v_\mathrm{N}$), east ($v_\mathrm{E}$), and up ($v_\mathrm{U}$) components of the Earth-fixed velocity vector of the tracking station (due to plate motion) are given by:

$$\begin{aligned}\frac{\partial \mathbf{r}_\mathrm{b}}{\partial v_\mathrm{N}} &= \frac{t - t_0}{3.15576 \times 10^{12}}\,\mathbf{N} \\ \frac{\partial \mathbf{r}_\mathrm{b}}{\partial v_\mathrm{E}} &= \frac{t - t_0}{3.15576 \times 10^{12}}\,\mathbf{E} \\ \frac{\partial \mathbf{r}_\mathrm{b}}{\partial v_\mathrm{U}} &= \frac{t - t_0}{3.15576 \times 10^{12}}\,\mathbf{Z}\end{aligned} \tag{5–210}$$

where $t$ and $t_0$ are the time argument and the user input epoch in seconds of coordinate time ET past J2000.

From Eqs. (5–1) and (5–13), the partial derivatives of $\mathbf{r}_\mathrm{b}$ with respect to the rectangular components of the Earth-fixed vector from the center of mass of the Earth to the origin for the input 1903.0 station coordinates are given by:

$$\begin{aligned}\frac{\partial \mathbf{r}_\mathrm{b}}{\partial x_\mathrm{O}} &= \begin{bmatrix}1\\0\\0\end{bmatrix} \\ \frac{\partial \mathbf{r}_\mathrm{b}}{\partial y_\mathrm{O}} &= \begin{bmatrix}0\\1\\0\end{bmatrix} \\ \frac{\partial \mathbf{r}_\mathrm{b}}{\partial z_\mathrm{O}} &= \begin{bmatrix}0\\0\\1\end{bmatrix}\end{aligned} \tag{5–211}$$

From Eqs. (5–1) and (5–22), the partial derivatives of $\mathbf{r}_\mathrm{b}$ with respect to constant corrections to the $X$ and $Y$ angular coordinates of the true pole of date relative to the mean pole of 1903.0 are given by:

$$\frac{\partial \mathbf{r}_b}{\partial X} = \begin{bmatrix} -z_b \\ 0 \\ x_b \end{bmatrix}$$

$$\frac{\partial \mathbf{r}_b}{\partial Y} = \begin{bmatrix} 0 \\ z_b \\ -y_b \end{bmatrix} \tag{5–212}$$

where $x_b$, $y_b$, and $z_b$ are rectangular components of the sum of the first four terms of Eq. (5–1). However, to sufficient accuracy, use the rectangular components of the first term of Eq. (5–1).

### 5.5.2 FRAME-TIE ROTATION ANGLES

From Eq. (5–113), the partial derivatives of the geocentric space-fixed position vector of the tracking station with respect to the frame-tie rotation angles $r_z$, $r_y$, and $r_x$ are given by:

$$\frac{\partial \mathbf{r}_{TS}^{E}}{\partial r_z} = \frac{\partial T_E}{\partial r_z} \mathbf{r}_b \qquad z \to y, x \tag{5–213}$$

where the partial derivatives of $T_E$ with respect to $r_z$, $r_y$, and $r_x$ are given by Eqs. (5–132) to (5–134), which use Eqs. (5–120) to (5–122).

### 5.5.3 UNIVERSAL TIME UT1

The partial derivative of the geocentric space-fixed position vector of the tracking station with respect to Universal Time UT1 is given by:

$$\frac{\partial \mathbf{r}_{TS}^{E}}{\partial \mathrm{UT1}} = \frac{\partial T_E}{\partial \mathrm{UT1}} \mathbf{r}_b \tag{5–214}$$

where the partial derivative on the right-hand side is given by Eqs. (5–135), (5–160), (5–18), (5–164), and (5–176). The vector $\mathbf{r}_b$ can be approximated by the first term of Eq. (5–1), which is evaluated using Eq. (5–2). Assembling all of these pieces and simplifying gives:

$$\frac{\partial \mathbf{r}_{\mathrm{TS}}^{\mathrm{E}}}{\partial \mathrm{UT1}} = \left(0.729{,}211{,}59 \times 10^{-4}\right)(\alpha u)\left[(NA)'\right]^{\mathrm{T}} \begin{bmatrix} -\sin(\theta+\lambda) \\ \cos(\theta+\lambda) \\ 0 \end{bmatrix} \tag{5–215}$$

where $(NA)'$ is given by Eq. (5–130), sidereal time $\theta$ is given by Eq. (5–162), and $u$ and $\lambda$ are input 1903.0 cylindrical station coordinates.